The Conformal Bootstrap

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The aim of this talk is to outline the methods of the conformal bootstrap.

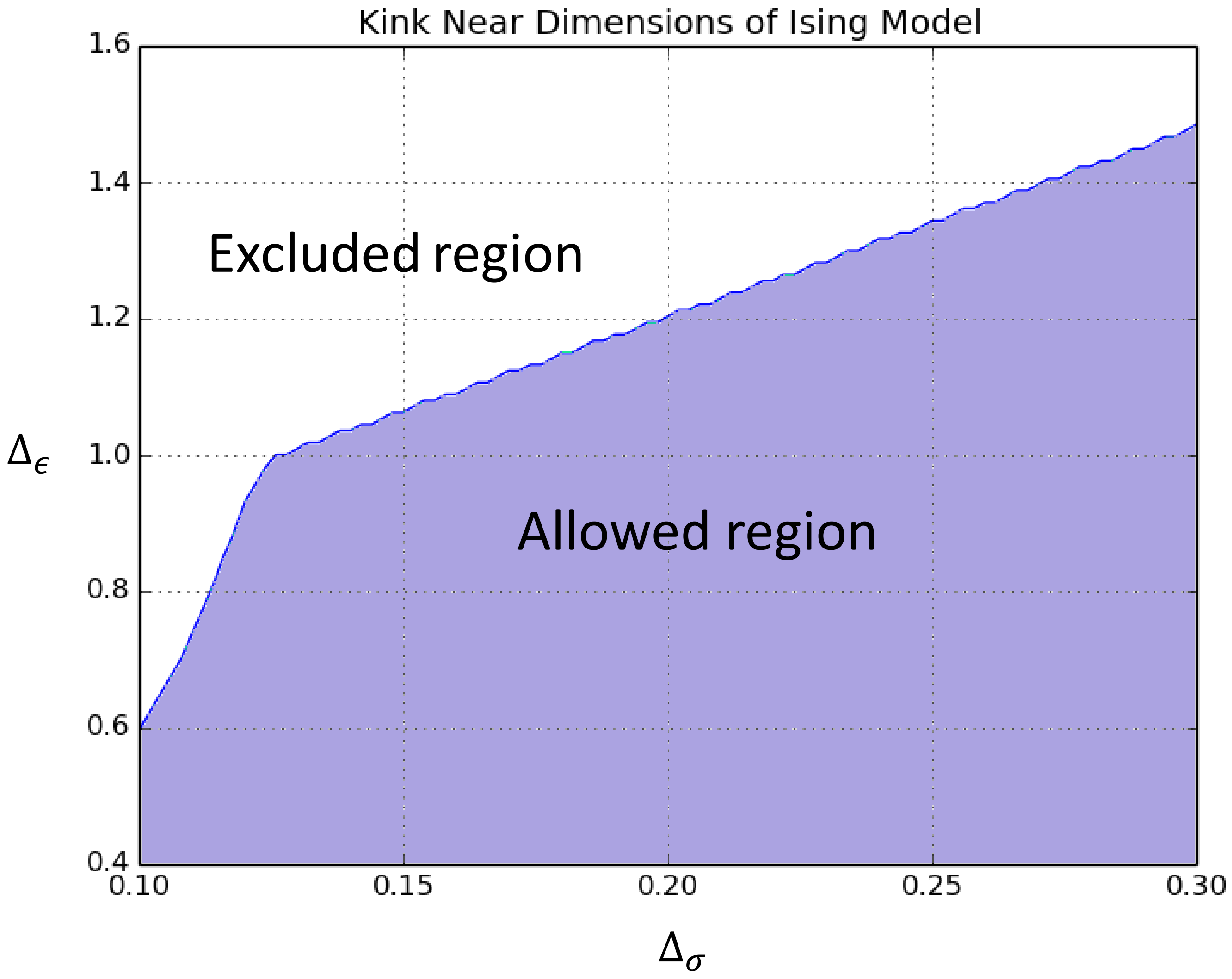
In the last decade the bootstrap has accelerated progress in exploring quantum field theories that describe physics with conformal symmetry, CFTs.

For example, this has resulted in the most precise predictions to date of the behaviour of physical quantities at certain phase transitions.

The bootstrap method constrains the space of possible CFTs.

- Numerical but non-perturbative.
- Applicable in great generality.

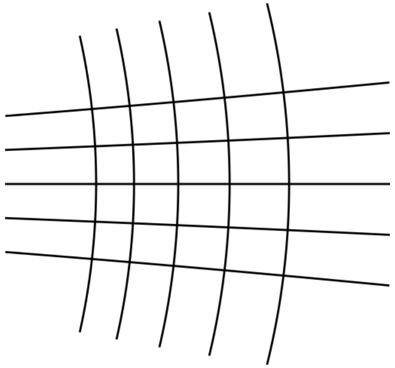
The plot shows a bound on the properties of the fields present in the 2D critical Ising model. We'll be a bit more precise later.



Equivalent plot for 3D due to El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi (2012)

Basics

Conformal transformations change the coordinates in a way that preserves angles.



Under these transformations, *primary operators* transform with a factor of the Jacobian. (Δ is the *scaling dimension* of ϕ)

$$\phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}} \phi(x)$$

Demanding covariance fixes the form of the 3-pt function.

$$\langle \phi_i(x)\phi_j(y)\phi_k(z)\rangle = \frac{C_{ijk}}{|x-y|^a|y-z|^b|z-x|^c}$$

The 4-pt is not so easy, but this is a good thing.

Let's just consider identical operators: $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$

Now have enough points to construct *conformal invariants*, on which we could have complicated dependence.

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$$

To progress, we expand products:

$$\phi(x_1)\phi(x_2) = \sum_\theta f_\theta(x_{12}, \partial_2)\theta(x_2)$$

The sum is over primary operators θ; it is called the operator product expansion, or OPE.

This turns an n-pt function into a sum of (n-1)-pt functions!

But which pairs to expand?

It shouldn't matter; we should get the same result regardless.

This is the consistency condition leveraged by the conformal bootstrap.

Main idea: demanding this mathematical consistency constrains the space of possible conformal quantum field theories.

$$\boxed{\text{Consistency conditions}} \xrightarrow[\text{D>2: 2008 on}]{\text{D=2: 1980's}} \boxed{\text{CFT data } (C_{ijk}, \Delta_{\mathrm{i}})} \xrightarrow{\text{1980's}} \boxed{\text{All correlators between fields}}$$

(Using the conformal algebra)

So, how do we do this?

The actual condition: $$\sum_{\theta \in \phi\phi\ OPE} C^2_{\phi\phi\theta}\left[v^{\Delta_\phi} g_{\Delta_\theta, l_\theta}(u,v) - u^{\Delta_\phi} g_{\Delta_\theta, l_\theta}(v,u)\right] = 0$$

Positive, real numbers
(we saw these in the 3-pt)

Known function, with
some unknown inputs.

- We control u, v and Δ_ϕ, and know the "conformal blocks" $g_{\Delta_\theta, l_\theta}(u,v)$.
- We have no idea which operators θ actually appear in $\phi\phi\ OPE$, so we don't know the dimensions (Δ_θ) and the spins (l_θ) to input into $g_{\Delta_\theta, l_\theta}(u,v)$.

Next, we guess.

Let's shrink the notation: $$\sum_{\theta \in \phi\phi\ OPE} C^2_{\phi\phi\theta} F^{\Delta_\phi}_{\Delta_\theta, l_\theta} = 0$$

We know F, even if we don't know what to input into F.

We also know C^2 is positive; this immediately means that given a CFT containing ϕ,

there must also be at least one primary operator θ such that $F^{\Delta_\phi}_{\Delta_\theta, l_\theta}$ is negative.

We have learned something!

$$\sum_{\theta \in \phi\phi\ OPE} C^2_{\phi\phi\theta} F^{\Delta_\phi}_{\Delta_\theta, l_\theta} = 0$$

That was a naïve example of how we could use this constraint, but the strategy is clear:

Certain assumptions about the spectrum of primary operators can be shown to be inconsistent with the above constraint, ruling out swathes of possible CFTs.

We still have freedom in choosing u and v; we can use this to create a vector of quantities which should all sum to zero with positive coefficients.

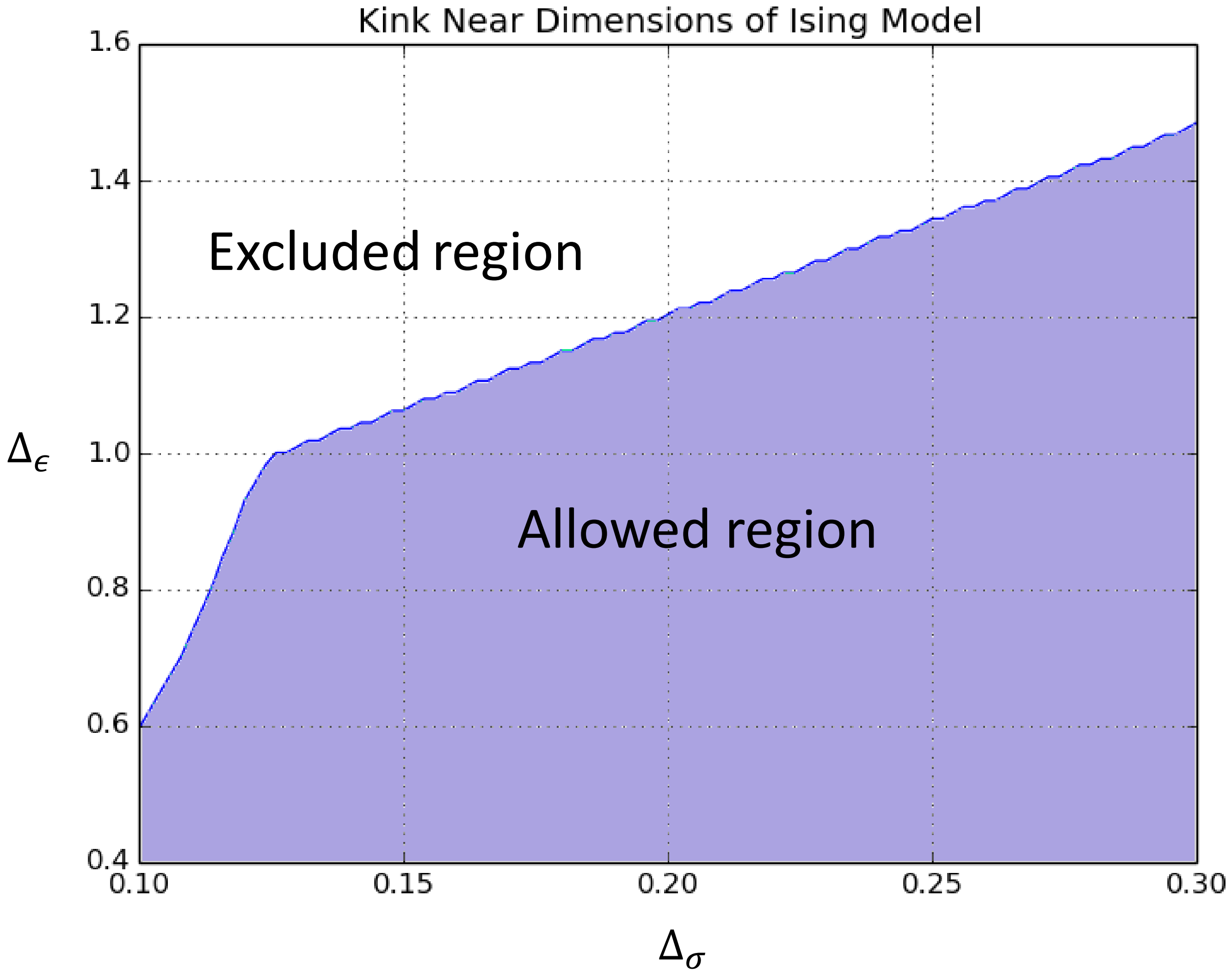
$$\sum_{\theta \in \phi\phi\ OPE} C^2_{\phi\phi\theta} \vec{G}^{\Delta_\phi}_{\Delta_\theta, l_\theta} = 0$$

This was the insight of Rattazzi, Rychkov, Tonni and Vichi in 2008.

This formulation translates the problem into one amenable to methods such as linear programming.

Undergrad project:

- understand theory
- calculate example of 4-pt correlator
- use JuliBootS software to generate bounds shown.



I am very grateful to Prof. Dmytro Volin for his guidance and enthusiasm throughout this project.

Equivalent plot for 3D due to El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi (2012)

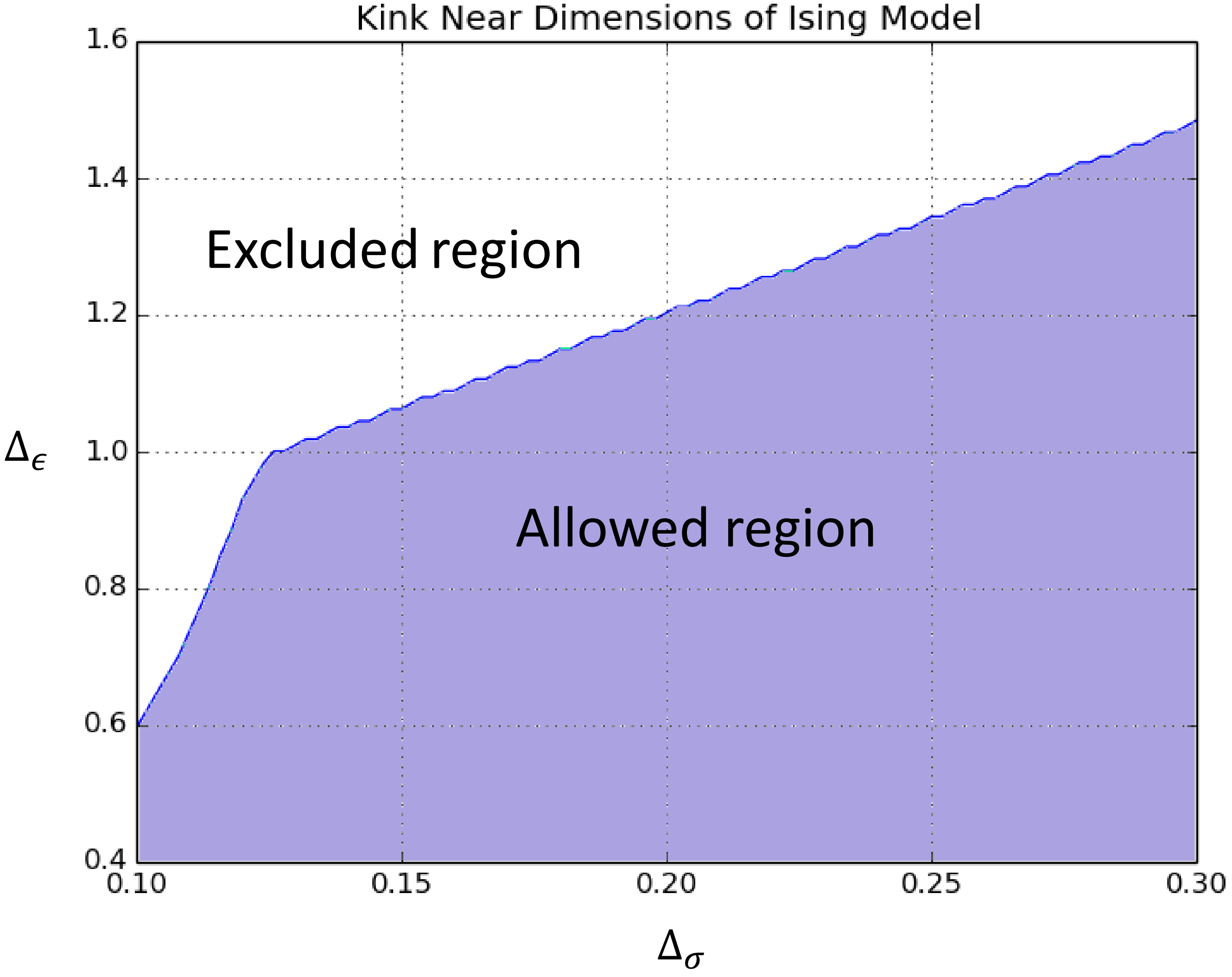
Using symmetry only found in 2D, one can classify the possible unitary QFTs with conformal symmetry.

For example, the exact 2D Ising critical exponents are known:

$$\Delta_\epsilon = 1$$
$$\Delta_\sigma = 0.125$$

Region shown was the bootstrap state of the art in 2012; modern constraints are much tighter.

Although, note the bound has already 'found' the Ising model.



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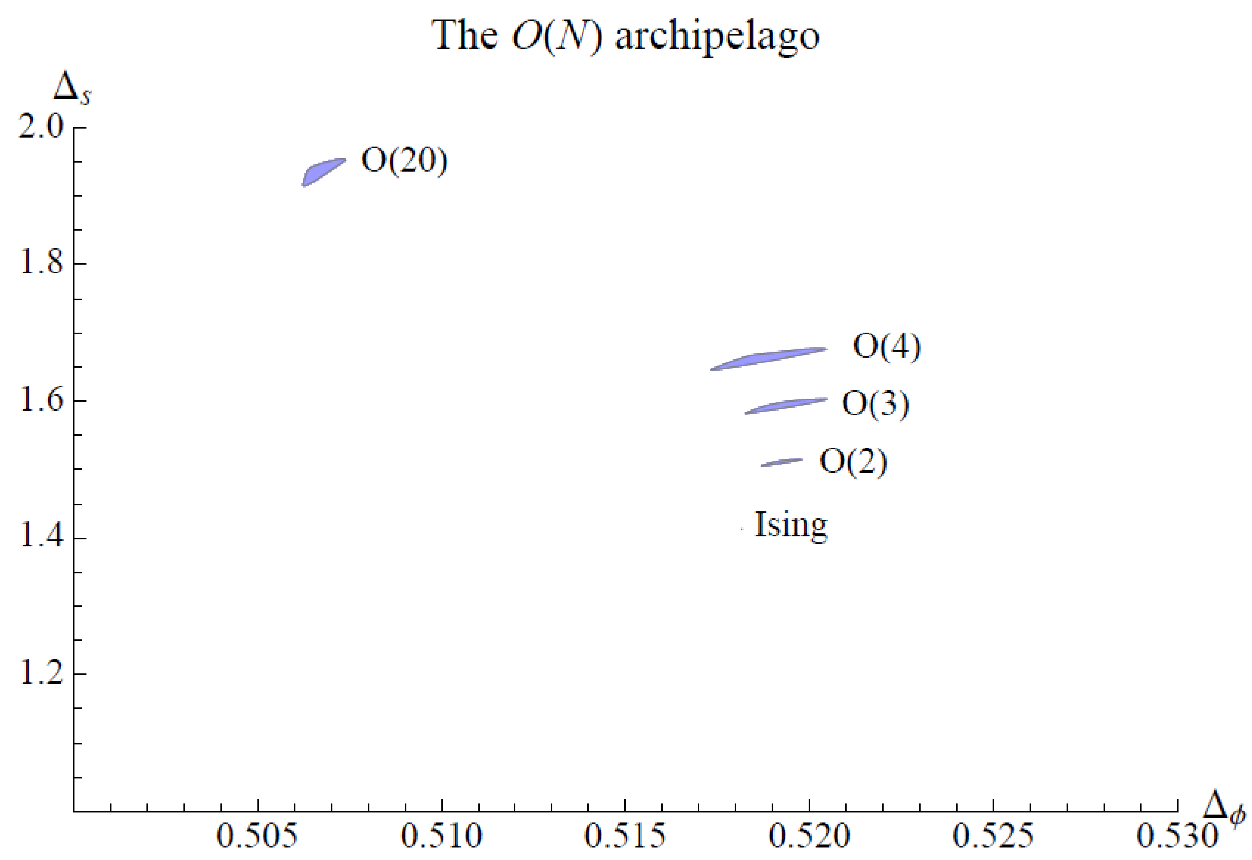
Improvements:

- Using correlators of non-identical operators.
- Better numerics.
- Taking further symmetry into account.



Bounds on scaling dimensions for 3D CFTs with $O(N)$ symmetry

Kosa, Poland, Simmons-Duffin, Vichi (2015)

Thank you!