Conformal Field Theory in Two Dimensions: Representation Theory and The Conformal Bootstrap

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Abstract

In this project we investigate quantum field theories in two dimensions which are invariant under conformal transformations. Conformal transformations are those which scale the metric by some local factor; demanding that a theory have these transformations as symmetries places powerful constraints on the form of the theory. We present the basic definitions and properties of conformal field theories, as well as the calculations needed to derive these constraints.

We present the exactly known space of possible theories (the minimal models). In light of the known result we present the conformal bootstrap, a more widely applicable method which imposes an associative algebraic structure on the fields of the theory, and from this obtains certain bounds on the field dimensions.

Contents

1 Conformal Theories

1.1 Motivation

Conformal theories are those which describe scale independent physics. Intuitively, this means the same phenomena are seen regardless of the scale of an observation. More rigorously, we insist that the physically measurable quantities in our theory (the correlators) must be expressed in a way that is invariant under certain transformations (called the conformal transformations). As it turns out, this is a powerful condition; for example, it immediately fixes the 2 and 3-point correlators. It is remarkable that this is done without any reference to a Lagrangian or a Hamiltonian; the only input is the symmetries of the theory, and some reference to the energy-momentum tensor. Two excellent references are the seminal work of Belavin, Polyakov and Zamolodchikov [1] and the notes of David Tong [6]; the first for a lucid and rigorous treatment, the second for a less rigorous but more intuitive overview.

These systems are not so rare as the stringent conditions may make them seem. A familiar example is the Ising model at its critical temperature (the temperature at which it loses long-range spin alignment). Here, the correlation length diverges, and the same fluctuations are seen at every length scale.

1.2 Conformal Symmetries

Let us be a bit more exact. Conformal field theories are those whose physics are invariant under the set of conformal transformations. A conformal transformation is one which changes the metric by scaling it by any function of the coordinates:

$$g_{\mu\nu}(x) \to \Lambda^2(x) g_{\mu\nu}(x) \tag{1}$$

The conformal symmetries form a group. This group is an extension of the Poincaré group (translations, rotations, boosts) by dilations and the special conformal transformations. The transformations of the Poincaré group leave the metric invariant, and so fulfill 1 trivially. The scale factor $\Lambda^2(x)$ is also easy to work out for dilations:

$$\begin{aligned} x^\mu &\to \lambda x^\mu \\ \implies g_{\mu\nu} &\to \lambda^{-2} g_{\mu\nu} \end{aligned}$$

For the special conformal transformations (SCTs) the scaling is less obvious. A special conformal transformation is defined as an inversion, followed by a translation (along a vector typically denoted b_μ), followed by another inversion. [1] The translation does not affect the metric, so by calculating the change in the metric produced by an inversion, we can find the change produced by the SCT. We find:

$$\begin{aligned} x^\mu &\to \frac{x^\mu}{x^2} \\ \implies g_{\mu\nu} &\to (\mathbf{x}^2)^2 g_{\mu\nu} \end{aligned}$$

[1] We use this as, unlike inversion, SCTs have a differentiable parameter.

	Finite form	Infinitesimal form	Generator
Translations	$x'^{\mu} = x^{\mu} + a^{\mu}$	$x'^{\mu} = x^{\mu} + \epsilon^{\mu}$	$P_{\mu} = -i\partial_{\mu}$
Rotations	$x'^{\mu} = \lambda x^{\mu}$	$x'^{\mu} = x^{\mu} + \lambda x^{\mu}$	$L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$
Dilations	$x'^{\mu} = M^{\mu}_{\nu} x^{\nu}$	$x'^{\mu} = x^{\mu} + \omega^{\mu}_{\nu} x^{\nu}$	$D = -ix^{\mu}\partial_{\mu}$
SCTs	$x'^{\mu} = \frac{x^{\mu} - b^{\mu}\mathbf{x}^2}{1 - 2\mathbf{b}\cdot\mathbf{x} + \mathbf{x}^2\mathbf{b}^2}$	$x'^{\mu} = x^{\mu} + b^{\mu}\mathbf{x}^2 - 2x^{\mu}\mathbf{b}\cdot\mathbf{x}$	$K_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - \mathbf{x}^2\partial_{\mu})$

Table 1: Forms of the conformal transformations

Thus the first inversion gives a factor of $(x^2)^2$, and the second a factor of the square of $(\frac{x^{\mu}}{x^2} - b^{\mu})(\frac{x_{\mu}}{x^2} - b_{\mu})$. Multiplying, we obtain the scale factor:

$$\Lambda^2(x) = (1 - 2\mathbf{b}\cdot\mathbf{x} + \mathbf{x}^2\mathbf{b}^2)^2$$

These generators obey the following commutation relations:

$$\begin{aligned} [D, P_{\mu}] &= iP_{\mu} \\ [D, K_{\mu}] &= -iK_{\mu} \\ [K_{\mu}, P_{\nu}] &= 2i(g_{\mu\nu}D - L_{\mu\nu}) \\ [K_{\rho}, L_{\mu\nu}] &= i(g_{\rho\mu}K_{\nu} - g_{\rho\nu}K_{\mu}) \\ [P_{\rho}, L_{\mu\nu}] &= i(g_{\rho\mu}P_{\nu} - g_{\rho\nu}P_{\mu}) \\ [L_{\rho\sigma}, L_{\mu\nu}] &= i(g_{\sigma\mu}L_{\rho\nu} - g_{\sigma\nu}L_{\rho\mu} + g_{\rho\mu}L_{\nu\sigma} - g_{\rho\nu}L_{\mu\sigma}) \end{aligned} \tag{2}$$

It is not a coincidence that these relations are reminiscent of the commutation relations for rotations; this is in fact isomorphic to $SO(d+1,1)$, though we will not use that fact here. See section 2.1 of [2] and section 4.1 of [3] for a more in-depth exploration of these topics, including derivations showing these are indeed the only globally defined conformal transformations.

1.3 Operator Product Expansion

Another hallmark of conformal field theories is the assumption of an operator product expansion. We start with some set of fields, known as the primary fields. We demand that these transform in a particular way (defined in 2.3). This demand ensures that we can fix the form of their correlators, as we will see in 4.2. Then, we assume that the product[2] of two fields can always be expanded in terms of local fields (we call this an OPE); this expansion will obey some algebra.

Say we take the expansion of a primary field ϕ with the energy-momentum tensor. We will obtain some sum of local fields. However, these fields will not be part of our set of primary fields (they will not transform correctly). We call these extra fields descendants of ϕ.

What of the expansion between two primary fields? This expansion can contain other

[2]Time ordered product, inside a correlator.

primary fields, as well as their descendants. If we have not already included these primary fields in our algebra, we must now include them. However, this means we must also include all the new primaries generated by expanding these additional fields with our original fields. At first it appears that the theory will get quite unwieldy very quickly, however there are two points that give us hope. Firstly, it is possible that this process of adding fields will not continue forever; we may find some finite set of fields for which this algebra is closed. Indeed, this turns out to be the case for a class of theories known as the minimal models (see end of 3.4). Secondly, as we will see in 5.4, the symmetries of the theory allow us to express the correlator of any descendant fields in terms of the correlator of the primary fields. As we have already fixed these, this means we have a chance at obtaining something very rare: an exactly solvable quantum field theory.

2 Two Dimensions: A Safari

2.1 Complex Coordinates

Before diving in, we present some necessary definitions. In the previous section, the conformal transformations discussed were all *global* conformal transformations. This means they are well behaved everywhere on the Riemann sphere. However, our physics will also be sensitive to transformations which are only locally defined. The algebra of these local conformal transformations will turn out be quite useful, due to the fact it is infinite dimensional. We will make extensive use of the powerful machinery of complex analysis to study this, relying heavily on tools such as Laurent expansions and the residue theorem.

First, some definitions:

$$\begin{aligned} z &= x + iy & x &= \frac{1}{2}(z + \bar{z}) \\ \bar{z} &= x - iy & y &= \frac{1}{2i}(z - \bar{z}) \\ \partial_z &= \frac{1}{2}(\partial_x - i\partial_y) & \partial_x &= \partial_z + \partial_{\bar{z}} \\ \partial_{\bar{z}} &= \frac{1}{2}(\partial_x + i\partial_y) & \partial_y &= i(\partial_z - \partial_{\bar{z}}) \end{aligned} \tag{3}$$

We consider z and $\bar{z}$ to be independent variables, only setting $\bar{z} = z^*$ at the end.

Transforming the metric we find:

$$\begin{aligned} g_{\mu\nu} &= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \\ g^{\mu\nu} &= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \end{aligned} \tag{4}$$

In these coordinates the Cauchy-Riemann equations greatly simplify:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\begin{aligned} &\iff \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0 \\ &\iff (\partial_x + i\partial_y)(u + iv) = 0 \\ &\iff \partial_{\bar{z}} f = 0 \end{aligned}$$

for $f(x, y) = u(x, y) + iv(x, y)$.

How do these coordinates relate to our familiar ones? We will deal with one time and one space dimension (x^0 and x^1), imposing periodic boundary conditions on the spatial coordinate. These are coordinates on the surface of a cylinder, with space being cyclic and time running along the axis. Our z and $\bar{z}$ are related to these by:

$$z = e^{x^0 + ix^1}$$
$$\bar{z} = e^{x^0 - ix^1}$$

This maps the cylinder to the complex plane, with the infinite past being mapped to the origin. Slices of constant time become circles, time ordering becomes radial ordering and the time evolution operator becomes the dilation operator.

2.2 The Energy-Momentum Tensor

We define the energy-momentum tensor as the variation of the action with respect to the metric

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

under an infinitesimal coordinate change $x'^{\mu} \to x^{\mu} + \epsilon^{\mu}$. This particular formulation is called the Belinfante-Rosenfeld energy-momentum tensor, and is automatically symmetric. We see that:

$$\delta S = \int d^2x T^{\mu\nu} \delta g_{\mu\nu}$$

For a conformal transformation the metric must be scaled, so $\delta g_{\mu\nu} = f(x) g_{\mu\nu}$

$$\delta S = \int d^2x T^{\mu\nu} f(x) g_{\mu\nu}$$
$$\delta S = \int d^2x T^{\mu}_{\mu} f(x)$$

If this arbitrary local conformal transformation is a symmetry of the theory, then $\delta S = 0$, so we obtain that the energy-momentum tensor of a conformal field theory is traceless.

We can use this to show something interesting. First, see that under an infinitesimal transformations $x'^{\mu} \to x^{\mu} + \epsilon^{\mu}$, the metric transforms as follows:

$$\begin{aligned} g'_{\mu\nu} &= g_{\alpha\beta}(\delta^{\alpha}_{\mu} + \partial_{\mu}\epsilon^{\alpha})(\delta^{\beta}_{\nu} + \partial_{\nu}\epsilon^{\beta}) \\ &\approx g_{\mu\nu} + \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} \end{aligned}$$

Now, consider the quantity $j^\mu = T^{\mu\nu}\epsilon_\nu$:

$$\begin{aligned}\partial_\mu j^\mu &= \partial_\mu (T^{\mu\nu})\epsilon_\nu + T^{\mu\nu}\partial_\mu\epsilon_\nu \\ &= T^{\mu\nu}\partial_\mu\epsilon_\nu \\ &= \frac{1}{2}T^{\mu\nu}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) \\ &= \frac{1}{2}T^{\mu\nu}(\delta g_{\mu\nu}) \\ &= \frac{1}{2}T^{\mu\nu}(f(x)g_{\mu\nu}) \\ &= \frac{1}{2}T^\mu_\mu f(x) \\ &= 0\end{aligned}$$

where we have used that the energy-momentum tensor is symmetric and conserved. We have shown that if this is a conformal transformation, then j^μ is the conserved current associated with it.

What does tracelessness mean in complex coordinates? We see:

$$T^\mu_\mu = g_{\mu\nu}T^{\mu\nu} = \frac{1}{2}T^{z\bar{z}} + \frac{1}{2}T^{\bar{z}z} = 0$$

Using the symmetry of the energy-momentum tensor, this implies that $T^{\bar{z}z} = 0$.
We can learn more about T using the fact that translation symmetry implies that it is conserved.
For $\nu = z$

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= 0 \\ \implies \partial_z T^{zz} + \partial_{\bar{z}} T^{\bar{z}z} &= 0 \\ \implies \qquad \partial_z T^{zz} &= 0 \\ \implies \qquad \partial_z T_{\bar{z}\bar{z}} &= 0\end{aligned}$$

Where we used the metric to lower indices to obtain that $T_{\bar{z}\bar{z}}$ is an antichiral field.
Similarly, for $\nu = \bar{z}$

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= 0 \\ \implies \partial_z T^{z\bar{z}} + \partial_{\bar{z}} T^{\bar{z}\bar{z}} &= 0 \\ \implies \qquad \partial_{\bar{z}} T^{\bar{z}\bar{z}} &= 0 \\ \implies \qquad \partial_{\bar{z}} T_{zz} &= 0\end{aligned} \tag{5}$$

we obtain that T_{zz} is a chiral field.
In light of this, we define $T(z) = T_{zz}(z, \bar{z})$ and $\bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}(z, \bar{z})$

2.3 Taxonomy of Fields

Here we categorise the fields which will populate our theory. While the physics must be invariant under conformal transformations, the fields themselves may not be; they merely have to form a representation of the symmetry algebra.

A *primary field* $\phi(z,\bar{z})$ is one which, under a conformal transformation $(z,\bar{z}) \mapsto (f(z), \bar{f}(\bar{z}))$, transforms as

$$\phi(z,\bar{z}) \mapsto \phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z}))$$

h and $\bar{h}$ are called the conformal weights of the field. All of the fields in our theory will either be primaries or descendants of primaries (that is, they appear in the Tϕ OPE, or the OPE of T with a descendant; note that this means that T itself is a descendant of the identity field, the vacuum). A primary and its descendants form a representation of the symmetry algebra, transforming among themselves under conformal transformations.

If a field transforms in the way defined above for only the global conformal transformations, it is called a *quasi-primary field*. We define a *chiral* (or *anti-chiral*) field as one which only depends on z (or $\bar{z}$).

The primary fields are the foundations of the theory; given the correlators between the primary fields, we can find the correlators of any of their descendants.

2.4 Holomorphic Functions are Conformal

A conformal transformation is one which scales the metric. Say we apply such a transformation to some metric:

$$\begin{aligned} x \to u &= u(x,y) \\ y \to v &= v(x,y) \\ g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \end{aligned}$$

From which we obtain the two conditions:

$$\begin{aligned} \frac{\partial x}{\partial u}^2 + \frac{\partial y}{\partial u}^2 &= \frac{\partial x}{\partial v}^2 + \frac{\partial y}{\partial v}^2 \\ \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} &= 0 \end{aligned}$$

We obtain two solutions:

$$\begin{aligned} \frac{\partial x}{\partial u} &= \pm\frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial v} &= \mp\frac{\partial y}{\partial u} \end{aligned}$$

We see that if we write $f(x,y) = u(x,y) + iv(x,y)$ the first solution becomes the Cauchy-Riemann Equations ($\partial_{\bar{z}} f = 0$) and the second becomes the equations defining antiholomorphicity ($\partial_z f = 0$). Thus, in two dimensions, we have a great freedom in choosing our transformation: all holomorphic transformations are conformal.

2.5 Symmetry Algebras

2.5.1 Witt Algebra

The Witt algebra is the algebra of the infinitesimal conformal transformations. While the global conformal transformations must be holomorphic on the entire Riemann sphere, the local conformal transformations need only be holomorphic on some open set; in general, they may be meromorphic.
Thus, any infinitesimal transformation of the coordinates can be expanded in a Laurent series to obtain:

$$
\begin{aligned}
z' &= z + \epsilon(z) = z + \sum_{n=-\infty}^{\infty} \epsilon_n(-z^{n+1}) \\
\bar{z}' &= \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{n=-\infty}^{\infty} \bar{\epsilon}_n(-\bar{z}^{n+1})
\end{aligned}
$$

for some constants ϵ_n and $\bar{\epsilon}_n$.
Motivated by this we define $l_n = -z^{n+1}\partial_z$ and $\bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$ for $n \in \mathbb{Z}$ as generators of these transformations. The commutator of these elements can then be seen to be:

$$
[l_m, l_n] = (m-n)l_{m+n} \tag{6}
$$

with the $\bar{l}_n$ having a similar commutator, and the two sets commuting with each other. This is known as the Witt algebra.
Note that for $n < -1$ the generators are not defined at $z = 0$. Similarly, at $z = \infty$, the generators corresponding to $n > 1$ are not defined; this can be seen by performing an inversion. Thus, the generators corresponding to the global conformal transformations are l_{-1}, l_0 and l_1.

What is the direct relation between these generators and the global conformal transformations? To determine this, let us switch to cartesian coordinates:

$$
\begin{aligned}
l_{-1} &= -\partial_z = -\frac{1}{2}(\partial_x - i\partial_y) \\
\bar{l}_{-1} &= -\partial_{\bar{z}} = -\frac{1}{2}(\partial_x + i\partial_y) \\
l_0 &= -z\partial_z = -\frac{1}{2}(x+iy)(\partial_x - i\partial_y) \\
\bar{l}_0 &= -\bar{z}\partial_{\bar{z}} = -\frac{1}{2}(x-iy)(\partial_x + i\partial_y) \\
l_1 &= -z^2\partial_z = -\frac{1}{2}(x+iy)^2(\partial_x - i\partial_y) = -\frac{1}{2}(x^2 + 2ixy - y^2)(\partial_x - i\partial_y) \\
\bar{l}_1 &= -\bar{z}^2\partial_{\bar{z}} = -\frac{1}{2}(x-iy)^2(\partial_x + i\partial_y) = -\frac{1}{2}(x^2 - 2ixy - y^2)(\partial_x + i\partial_y)
\end{aligned}
$$

Comparing this to 1 we find the relations:

$$\begin{aligned} P_x &= i(\bar{l}_{-1} + l_{-1}) \\ P_y &= \bar{l}_{-1} - l_{-1} \\ L_{xy} &= l_0 - \bar{l}_0 \\ D &= -i(l_0 + \bar{l}_0) \\ K_x &= i(l_1 + \bar{l}_1) \\ K_y &= l_1 - \bar{l}_1 \end{aligned} \tag{7}$$

2.5.2 Virasoro Algebra

The generators of a symmetry act on operators through their adjoint representation:

$$[l_n, \phi] = \delta_n \phi$$

We know that these δ_n must obey the Witt algebra; they should combine in the same way the conformal transformations themselves do. However, this does not mean that the generators of the symmetry also obey this same algebra. All we know is that:

$$[ad_{l_m}, ad_{l_n}] = (m-n) ad_{l_{m+n}}$$

The l_n may obey the Witt algebra, or they may obey it up to the addition of a term that commutes with all other elements of the algebra. Allowing for this term results in an algebra known as the Virasoro algebra.
The Virasoro algebra has elements L_n for $n \in \mathbb{Z}$. The commutator of these elements is defined by:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{8}$$

This is a central extension to the Witt algebra. It is in fact the only non-trivial central extension; the added term is the only way to add a central term (one which commutes with all other elements of the algebra) while retaining bilinearity and the Jacobi identity as properties of the commutator. c is known as the central charge, with the factor $\frac{1}{12}$ being conventional. We shall see later that the central extension is very important; the only conformal field theory with $c = 0$ is the one containing only the vacuum state.

For $n \in \{-1, 0, 1\}$ we obtain a simpler subalgebra:

$$\begin{aligned} [L_0, L_1] &= -L_1 \\ [L_1, L_{-1}] &= 2L_0 \\ [L_{-1}, L_0] &= -L_{-1} \end{aligned} \tag{9}$$

Note that the central extension does not come into play here, so 7 is still valid. In two dimensions the algebra of global conformal transformations is isomorphic to the direct sum of two copies of this subalgebra. This algebra is $sl(2, \mathbb{R})$, the same algebra obeyed by J_z, J_+, J_-, the angular momentum operators. We use $\bar{L}_n$ to denote generators from the second copy.

3 Representation Theory of the Virasoro Algebra

3.1 Highest Weight Representations

The energy eigenstates of a conformal field theory form a representation of the Virasoro algebra. We will study what is known as the *highest weight* representation. This involves using physical constraints to argue the existence of a state highest conformal weight (lowest energy). Other states in the representation (the *descendants* of this state) are found by acting on it with the L_n.

In the quantization of angular momentum, the eigenstates of J_z span the representation space and J_+, J_- are used as raising and lowering operators. Then, by imposing that our states must have non-negative norm (this condition is known as *unitarity*) we obtain the well-known result that for this to be a sensible theory of angular momentum, j, the highest eigenvalue of J_z, must be an integer or half-integer, and that a complete set of states is given by applying J_- at most $2j$ times.

We shall do something similar here in our search for a representation of the Virasoro algebra. L_0 will act as our J_z; we shall use the eigenstates of this operator as a basis for the space, by choosing it to be diagonal. As can be seen from the commutation relation:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

none of the L_n commute with each other, so this is the only operator we can diagonalise. Instead of just having J_+ and J_-, the Virasoro algebra gives us a plethora of raising and lowering operators to play with. Consider some state $|m\rangle$ with eigenvalue m:

$$\begin{aligned} L_0\left(L_k|m\rangle\right) &= \left(L_k L_0 + [L_0, L_k]\right)|m\rangle \\ &= \left(L_k L_0 + (0-k)L_{0+k}\right)|m\rangle \\ &= \left(L_k m - kL_k\right)|m\rangle \\ &= (m-k)\, L_k|m\rangle \end{aligned}$$

For $n > 0$, the L_n act to create states with lower eigenvalues, while the L_{-n} act to create states with higher eigenvalues.

3.2 Unitarity Constraints

In 4.2 we will see that these eigenvalues come into the correlator between two fields. If we were allowed to act to create a state with a negative eigenvalue, this would result in the 2-point function of that field increasing without bound at large distances. Thus, to ensure a sensible physical interpretation, we must assume the existence of a state with smallest eigenvalue; conventionally this is called the *highest weight state* or *primary state* and is denoted by $|h\rangle$. This is defined by $L_0|h\rangle = h$ and:

$$L_n|h\rangle = 0, \quad n > 0$$

We also define a *quasi-primary state* as any state that vanishes when acted upon by L_1. A descendant of a primary state, created by acting on it with $L_{-k_1}L_{-k_2}...L_{-k_p}$, say, will

have eigenvalue $h + k_1 + k_2 + ... + k_p$.
We can also define an inner product in this space by setting $L_p^\dagger = L_{-p}$. For example, let us calculate the norm of $L_{-1}|h\rangle$:

$$\begin{aligned} ||L_{-1}|h\rangle|| &= \langle h|L_1 L_{-1}|h\rangle \\ &= \langle h|L_{-1}L_1 + 2L_0|h\rangle \\ &= 2h\langle h|h\rangle \end{aligned}$$

Where we have used the commutator and defined $(|h\rangle)^\dagger = \langle h|$. For this to have a sensible physical interpretation, we must therefore have that h is positive real. The vacuum state has $h = 0$; it is annihilated by L_0, so invariant under the time evolution operator.

Let's try and obtain a condition on c, by finding the norm of $L_{-p}|h\rangle$.

$$\begin{aligned} ||L_{-p}|h\rangle|| &= \langle h|L_p L_{-p}|h\rangle \\ &= \langle h|L_{-p}L_p + 2pL_0 + \frac{c}{12}(p^3 - p)|h\rangle \\ &= p\left(2h + \frac{c}{12}(p^2 - 1)\right)\langle h|h\rangle \end{aligned}$$

If c is negative, then no matter what h is, we can always get a large enough p such that the norm is negative. Thus, c must be positive.

It is obvious that this is a powerful method for constraining the values of h and c which correspond to a physically sensible theory; with minimal effort and taking only two simple examples we have already obtained two such constraints.

3.3 Null Vectors

The conditions defining a highest weight state demand that it is annihilated by all raising operators. In that case, the descendants of that state form a representation of the Virasoro algebra. Is this representation irreducible? It is possible that some linear combination of descendants of the highest weight state might also be annihilated by all raising operators. For example, consider the state:

$$|\chi\rangle = (\alpha L_{-2} + L_{-1}^2)\,|h\rangle$$

We want to find α such that this vector is annihilated by all L_n with $n > 0$. It is sufficient to show it is annihilated by L_1 and L_2; using the Virasoro algebra we can see this from $L_n = \frac{1}{n-2}\left(L_{n-1}L_1 - L_1 L_{n-1}\right)$.
Applying L_1:

$$\begin{aligned} L_1\,|\chi\rangle &= L_1(\alpha L_{-2} + L_{-1}^2)\,|h\rangle \\ &= (\alpha(L_{-2}L_1 + 3L_{-1}) + (L_{-1}L_1 + 2L_0)L_{-1})\,|h\rangle \\ &= (\alpha(0 + 3L_{-1}) + L_{-1}(L_{-1}L_1 + 2L_0) + 2(L_{-1}L_0 + L_{-1}))\,|h\rangle \\ &= (\alpha(3L_{-1}) + L_{-1}(0 + 2h) + 2(L_{-1}h + L_{-1}))\,|h\rangle \\ &= (3\alpha + 4h + 2)\,|h\rangle \end{aligned}$$

So we must have that $\alpha = -\frac{4h+2}{3}$.
Applying L_2:

$$\begin{aligned} L_2 \left|\chi\right\rangle &= L_2(\alpha L_{-2} + L_{-1}^2) \left|h\right\rangle \\ &= \left(\alpha \left(L_{-2}L_2 + 4L_0 + \frac{c}{12}(6)\right) + (L_{-1}L_2 + 3L_1)L_{-1}\right) \left|h\right\rangle \\ &= \left(\alpha \left(4h + \frac{c}{2}\right) + 6h\right) \left|h\right\rangle \end{aligned}$$

For this to vanish, we must have:

$$h = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right) \tag{10}$$

So, if we have a theory with c and h obeying this relation, then $\left|\chi\right\rangle$ is a new highest weight state; its descendents will form another representation of the Virasoro algebra. We call this a null vector. If we want our representation to be irreducible, we must quotient out this subrepresentation; that is, if two states differ by a null vector or a descendent of a null vector, we say they are equal. We do this by setting all the null vectors in our algebra to 0. Can we do this consistently? If two states differ by only a null vector, do their inner products with other states match? The inner product of a null vector with a state of different eigenvalue will be zero, so we need only consider states at the same level. Let $\left|\chi\right\rangle$ be a null vector and $\left|\phi\right\rangle$ be a state at the same level. Then $\left\langle\phi\right|$ will be $\left\langle h\right|$ multiplied on the right by some combination of L_n with $n > 0$. Thus, the product will vanish. Note that this also implies the norm of $\left|\chi\right\rangle$ is zero. As a result of this we can set all null vectors and their descendants to zero, seriously reducing the number of elements in our algebra.

3.4 Kac Determinant

We can consider these inner products in more general terms. The inner product of two states with different eigenvalues will always be zero, so we will learn nothing new from considering these. If we consider the matrix M of all possible inner products, arranged in increasing order of eigenvalue, it will thus be block diagonal. It must also be Hermitian, as we impose that $\langle a|b\rangle$ is always real. We can therefore find a unitary matrix U that will diagonalise M. The diagonal elements of this matrix are the eigenvalues of M. Let $D = U^\dagger M U$. If one of the eigenvalues is negative, we can easily find a $\vec{b}$ such that:

$$\begin{aligned} & \vec{b}^t D \vec{b} && < 0 \\ \iff & \vec{b}^t U^\dagger M U \vec{b} && < 0 \\ \iff & \left(U\vec{b}\right)^\dagger M \left(U\vec{b}\right) && < 0 \end{aligned}$$

so the components of $U\vec{b}$ will give us the coefficients of a linear combination of states that will have negative norm. To avoid this, we must impose that no such negative eigenvalue exists. This implies that the determinant, as the product of the eigenvalues, must be positive.

Luckily, there exists a general formula for the determinant of each block of M (which we will denote by $M^{(l)}$):

$$detM^{(l)} = \alpha_l \prod_{rs \leq l} (h - h_{r,s}(c))^{p[l-rs]} \tag{11}$$

where $r, s \geq 1$ are integers, $p[n]$ is the partition function (0 if $n < 0$) and α_l is some known positive constant. This is known as the Kac determinant.

It is usual to parametrise $h_{r,s}(c)$ by parameter m:

$$h_{r,s}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}$$
$$c(m) = 1 - \frac{6}{m(m+1)}$$

Here m is defined implicitly in terms of c. However, if we write $h_{r,s}(c)$ explicitly:

$$h_{r,s}(c) = \frac{24(r-s)^2 + (c-1)(4 - 2(r^2+s^2)) + 2(r^2 - s^2)\sqrt{(1-c)(25-c))}}{96}$$

we notice that 10 matches this exactly, for $r = 2$ and $s = 1$, or $r = 1$ and $s = 2$. This should not surprise us; the values at which the Kac determinant vanishes are exactly those at which there exists a state of zero norm.

The entire region with $h > 0$ and $c > 1$ is unitary. This can be shown using the Kac determinant, but we will not do so here; see section 7.2.2 of [3] for a proof. Instead, we will concentrate on the region $0 < c < 1$. While finding conditions that are sufficient for unitarity in this region is difficult, we can (following section 7.2.3 of [3]) at least argue for some which are necessary. Each $h_{r,s}$ consists of two branches (for each sign of the square root) which meet at $c = 1$.

$$h_{r,s}(1) = \frac{(r-s)^2}{4}$$

As $p[0] = 1$, each factor of $h - h_{r,s}$ appears only linearly when $rs = l$. This means that if we vary the value of c over this line, the Kac determinant will change sign, and we will reach a region with values of h, c that do not correspond to unitary theories. Taking c very close to 1, and so keeping only the lowest order terms in $c - 1$:

$$h_{r,s}(c) \approx \frac{24(r-s)^2 \pm 2(r^2 - s^2)\sqrt{24(1-c)}}{96}$$

Suppose we keep $r - s$ fixed, but increase the product rs. We rewrite the above as:

$$h_{r,s}(c) \approx \frac{24(r-s)^2 \pm 2(r-s)\sqrt{(r-s)^2 + 4rs}\sqrt{24(1-c)}}{96}$$

It is now clear that for c very close to 1, the line $h_{r,s}(c)$ (for a given $r - s$) will get closer and closer to vertical as rs increases. This $h_{r,s}$ will first become relevant to the Kac determinant at $l = rs$, where it will appear linearly. Thus, by starting in the region $c > 1$ and reducing c until it crosses into $c < 1$, we can always find a level at which our path will be forced to cross a curve appearing linearly in the determinant, so causing the Kac determinant to become negative. The only points which escape this argument are those which end up lying on these curves. Here the determinant is zero, so we cannot easily extract whether or not these are unitary.

It can be shown that the only positive values of (c, h) with c strictly less than one that are unitary are those which correspond to $h_{r,s}$ and c above with *integer* $m \geq 2$. We also now restrict $1 \leq r < m$ and $1 \leq s \leq r$. These are known as the minimal models.
It is remarkable that the possible conformal dimensions are restricted in this way. One result we can immeditely note is that any conformal field theory with $c = 0$ contains only the vacuum state. For $c = 0$, the above formula gives $m = 2$; this implies that $r = 1$ and thus $s = 1$, so there is only one possible field. Seeing that $h_{1,1} = 0$ reveals this state as the vaccum state.

The critical Ising model is described by the minimal model with $m = 3$. The Ising model has three fields; the identity field I (the vacuum), the spin field σ and the energy field ϵ. From the above, we see $c = \frac{1}{2}$ and:

$$\begin{aligned} h_{1,1} &= 0 \\ h_{2,1} &= \frac{1}{2} \\ h_{2,2} &= \frac{1}{16} \end{aligned}$$

are the conformal dimensions of the three fields.

4 Constraining the Correlators

4.1 Using the Symmetries

How do the L_n act on the states in our theory? The states we defined above are actually the boundary conditions of our theory. If we have some operator $\hat{h}$ that acts on the vacuum to create a state $\hat{h}(z,\bar{z})\left|0\right\rangle$, then $\left|h\right\rangle$ is the limit of this state as $z,\bar{z}$ both go to zero. Similarly, $\left\langle h\right|$ is the limit of the conjugate of this state as $z,\bar{z}$ both go to ∞. These are asymptotic states; imagine some interaction between particles say, free in the infinite past, interacting for a period, then free again in the infinite future.
Since we know how the L_n act on these asymptotic states we can determine how they act on states away from the origin by using a translation in z (or, similarly, $\bar{z}$).

Consider, for brevity, a chiral state $\left|\phi(z)\right\rangle$ of conformal dimension h.
Using $[L_0, L_{-1}] = L_{-1}$:

$$\begin{aligned}
L_0\left|\phi(z)\right\rangle &= L_0 e^{zL_{-1}}\left|\phi(0)\right\rangle \\
&= L_0\left(\sum_{n=0}^{\infty}\frac{1}{n!}z^n L_{-1}^n\right)\left|\phi(0)\right\rangle \\
&= \left(\sum_{n=0}^{\infty}\frac{1}{n!}z^n L_{-1}^n\left(L_0+n\right)\right)\left|\phi(0)\right\rangle \\
&= \left(e^{zL_{-1}}L_0 + zL_{-1}\sum_{n=1}^{\infty}\frac{1}{(n-1)!}z^{n-1}L_{-1}^{n-1}\right)\left|\phi(0)\right\rangle \\
&= \left(e^{zL_{-1}}L_0 + zL_{-1}e^{zL_{-1}}\right)\left|\phi(0)\right\rangle \\
&= \left(h + z\partial_z\right)\left|\phi(z)\right\rangle
\end{aligned}$$

Using $[L_1, L_{-1}] = 2L_0$:

$$\begin{aligned}
L_1\left|\phi(z)\right\rangle &= L_1 e^{zL_{-1}}\left|\phi(0)\right\rangle \\
&= L_1\left(\sum_{n=0}^{\infty}\frac{1}{n!}z^n L_{-1}^n\right)\left|\phi(0)\right\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}z^n\left(L_{-1}^n L_1 + 2nL_{-1}^{n-1}L_0 + (n)(n-1)L_{-1}^{n-1}\right)\left|\phi(0)\right\rangle \\
&= \left(e^{zL_{-1}}L_1 + 2z\sum_{n=1}^{\infty}\frac{1}{(n-1)!}z^{n-1}L_{-1}^{n-1}L_0 + z^2\sum_{n=2}^{\infty}\frac{1}{(n-2)!}z^{n-2}L_{-1}^{n-1}\right)\left|\phi(0)\right\rangle \\
&= \left(e^{zL_{-1}}L_1 + 2ze^{zL_{-1}}L_0 + z^2e^{zL_{-1}}L_{-1}\right)\left|\phi(0)\right\rangle \\
&= \left(0 + 2zh + z^2\partial_z\right)\left|\phi(z)\right\rangle \\
&= \left(z^2\partial_z + 2zh\right)\left|\phi(z)\right\rangle
\end{aligned}$$

In the previous two calculations we commuted powers of L_n using relations easily verified by induction. We also used that $L_1\left|\phi(0)\right\rangle = 0$, as $\left|\phi(0)\right\rangle$ is a highest weight state.

So for some correlator f of many fields ϕ_i (dimension h_i) at position z_i we have (defining $\partial_i = \frac{\partial}{\partial z_i}$):

$$\begin{aligned} L_{-1}f &= 0 = \sum_i \partial_i f \\ L_0 f &= 0 = \sum_i (z_i \partial_i + h_i) f \\ L_1 f &= 0 = \sum_i (z_i^2 \partial_i + 2h_i z_i) f \end{aligned} \tag{12}$$

as the action of these generators must be zero if f is a correlator. Thus, we have three differential equations the correlators must obey.

We can make some general statements here about the first and second equations. Under a change of coordinates:

$$\begin{aligned} w_1 &= z_1 \\ w_2 &= z_2 - z_1 \\ w_3 &= z_3 - z_1 \\ &\dots \\ w_n &= z_n - z_1 \end{aligned}$$

we see that the first equation becomes $\frac{\partial f}{\partial w_1} = 0$; f can only depend on the differences between the z_i (translational invariance). Motivated by this, define $z_{ij} = z_i - z_j$.
The second equation ensures the correct scaling of f; it essentially shows the effect of the Euler operator on the correlator.

4.2 Fixing the 2 and 3-point Correlators

4.2.1 The 2-point correlator

We want to fix the form of $f(z_i, z_j) = \langle \phi_i(z_i) \phi_j(z_j) \rangle$.
The above equations 12 take the form:

$$\begin{aligned} L_{-1}f &= 0 = (\partial_i + \partial_j) f \\ L_0 f &= 0 = (z_i \partial_i + h_i + z_j \partial_j + h_j) f \\ L_1 f &= 0 = (z_i^2 \partial_i + 2h_i z_i + z_j^2 \partial_j + 2h_j z_j) f \end{aligned}$$

As shown, the first eqaution implies $f(z_i, z_j) = f(z_{ij})$.
Again using the first equation, the second simplifies to:

$$\begin{aligned} (z_i - z_j)\partial_i f &= -(h_i + h_j) f \\ \implies f(z_{ij}) &= \frac{d_{ij}}{z_{ij}^{h_i + h_j}} \end{aligned}$$

for some constant d_{ij}.
Finally, using the third equation, we obtain a relation between h_i and h_j. In the following we use the first equation, and that $(z_i - z_j)\partial_i f = -(h_i + h_j)f$.

$$\begin{aligned}
& (z_i^2\partial_i + 2h_i z_i + z_j^2\partial_j + 2h_j z_j)f = 0 \\
\implies & (z_i^2\partial_i + 2h_i z_i - z_j^2\partial_i + 2h_j z_j)f = 0 \\
\implies & (z_i^2 - z_j^2)\partial_i f = -2(h_i z_i + h_j z_j)f \\
\implies & (z_i + z_j)(-h_i - h_j)f = -2(h_i z_i + h_j z_j)f \\
\implies & z_i h_j + z_j h_i = h_i z_i + h_j z_j \\
\implies & z_i(h_j - h_i) = (h_j - h_i)z_j \\
\implies & h_j = h_i
\end{aligned}$$

We see that either $f = 0$ or $h_i = h_j$.
This gives our final result for the 2-point correlator:

$$\langle \phi_i(z_i)\phi_j(z_j)\rangle = \frac{d_{ij}\delta_{h_i,h_j}}{(z_1 - z_2)^{2h_i}} \tag{13}$$

where the d_{ij} are called structure constants. We will choose the basis of our fields so that $d_{ij} = 1$

4.2.2 The 3-point correlator

Here we want to fix the form of $f(z_i, z_j, z_k) = \langle \phi_i(z_i)\phi_j(z_j)\phi_k(z_k)\rangle$.
By translation invariance (the first equation), we have again that f can only depend on the differences between the positions. By dilation invariance (the second equation) we obtain:

$$(z_i\partial_i + z_j\partial_j + z_k\partial_k + h_i + h_j + h_k)f = 0$$

Now we can change coordinates to z_{ij}, z_{jk}, z_{ki} (note this is not invertible), so:

$$\begin{aligned}
\partial_i &= \partial_{ij} - \partial_{ki} \\
\partial_j &= \partial_{jk} - \partial_{ij} \\
\partial_k &= \partial_{ki} - \partial_{jk}
\end{aligned}$$

We obtain:

$$(z_{ij}\partial_{ij} + z_{jk}\partial_{jk} + z_{ki}\partial_{ki} + h_i + h_j + h_k)f(z_{ij}, z_{jk}, z_{ki}) = 0$$

The solution of this is

$$f(z_{ij}, z_{jk}, z_{ki}) = \frac{C_{ijk}}{z_{ij}^a z_{jk}^b z_{ki}^c}$$

with theory dependent structure constants C_{ijk}, and $a+b+c = h_i + h_j + h_k$. We could also have a sum of such functions, with different values of a, b, c, but as we shall see this is prevented by our third condition, SCT invariance.
Instead of using the third differential equation, we shall directly use that f must be invariant under the transformation $z \mapsto -\frac{1}{z}$. This implies our fields transform with a factor of $\left(\frac{1}{z^2}\right)^h$, and that $z_{12} \mapsto \frac{z_{12}}{z_1 z_2}$. To impose invariance, we must have:

$$\begin{aligned}
\langle \phi_i(z_i)\phi_j(z_j)\phi_k(z_k)\rangle &= \left(\frac{1}{z_i^2}\right)^{h_i}\left(\frac{1}{z_j^2}\right)^{h_j}\left(\frac{1}{z_k^2}\right)^{h_k}\langle \phi_i\left(-\frac{1}{z_i}\right)\phi_j\left(-\frac{1}{z_j}\right)\phi_k\left(-\frac{1}{z_k}\right)\rangle \\
\implies \qquad f(z_i, z_j, z_k) &= \left(\frac{1}{z_i^2}\right)^{h_i}\left(\frac{1}{z_j^2}\right)^{h_j}\left(\frac{1}{z_k^2}\right)^{h_k} f\left(-\frac{1}{z_i}, -\frac{1}{z_j}, -\frac{1}{z_k}\right) \\
\implies \qquad \frac{1}{z_{ij}^a z_{jk}^b z_{ki}^c} &= \frac{1}{z_i^{2h_i} z_j^{2h_j} z_k^{2h_k}} \frac{1}{z_{ij}^a z_{jk}^b z_{ki}^c}(z_i z_j)^a (z_j z_k)^b (z_k z_i)^c \\
\implies \qquad 1 &= z_i^{a+c-2h_i} z_j^{a+b-2h_j} z_k^{b+c-2h_k} \\
\implies \qquad a + c &= 2h_i \\
a + b &= 2h_j \\
b + c &= 2h_k
\end{aligned}$$

From which we obtain:

$$\begin{aligned}
a &= h_1 + h_2 - h_3 \\
b &= h_2 + h_3 - h_1 \\
c &= h_1 + h_3 - h_2
\end{aligned}$$

4.3 Failing to Fix the 4-Point Correlator

After our success in fixing the 2 and 3-point correlators, one would naturally hope to fix the 4-point in a similar fashion. Unfortunately, any such attempt is doomed to failure, due to the existence of the so-called cross-ratios:

$$\begin{aligned}
u &= \frac{z_{12} z_{34}}{z_{13} z_{24}} \\
\bar{u} &= \frac{\bar{z}_{12} \bar{z}_{34}}{\bar{z}_{13} \bar{z}_{24}}
\end{aligned}$$

These ratios are manifestly translation, dilation and rotation invariant. It can also easily be shown that they are invariant under inversions. Thus, the 4-point correlator can depend on these ratios in an arbitrary way. The arguments we have used thus far are insufficient.

5 The Operator Product Expansion

5.1 Terms of Use

An operator product expansion (OPE) is a very useful axiom of conformal field theory. We assume that inside a correlator, the product of two fields (at z and w say) can be written as a Laurent expansion in $z - w$, convergent in some punctured disk; the radius of convergence is equal to the distance to the nearest other operator in the correlator. The coefficients are local fields. The OPE can include $log(z)$ terms, but in that case it is generally more useful to work with the derivative of that OPE, as this often separates the OPE into holomorphic and anti-holomorphic parts.

5.2 The $T\phi$ OPE and the Ward Identity

Following section 2.5 of [2] we will find the OPE of the energy-momentum tensor with any primary field. Recall that a chiral primary field is one such that under any conformal transformation $z \mapsto f(z)$:

$$\phi(z) \mapsto \phi'(z) = \left(\frac{\partial f}{\partial z}\right)^h \phi(f(z))$$

Say $f(z) = z + \epsilon(z)$, with ϵ small. Then $\phi(z + \epsilon(z)) \approx \phi(z) + \epsilon\partial_z\phi$ and $\frac{\partial f}{\partial z} \approx 1 + \partial_z\epsilon$. As a result:

$$\begin{aligned}\phi(z) \mapsto \phi'(z) &\approx (\phi(z) + \epsilon\partial_z\phi)(1 + h\partial_z\epsilon) \\ &\approx \phi(z) + \epsilon\partial_z\phi + h\phi(z)\partial_z\epsilon\end{aligned}$$

If ϕ had a $\bar{z}$ dependence we would add a similar set of terms involving $\bar{\epsilon}$.

Recall that $j^\mu = T^{\mu\nu}\epsilon_\nu$ is a conserved current. We can obtain a conserved charge by integrating over a slice of constant time. In our coordinates, this amounts to doing a contour integral over a circle. Let's investigate the variation of some chiral field ϕ under a transformation generated by a conserved charge Q:

$$Q = \frac{1}{2\pi i}\oint dz T(z)\epsilon(z)$$

As usual, for a field with $\bar{z}$ dependence we would have another term.
The variation of ϕ under this transformation takes the form:

$$\begin{aligned}\delta\phi(w) &= [Q, \phi] \\ &= \frac{1}{2\pi i}\oint dz[T(z)\epsilon(z), \phi(w)] \\ &= \frac{1}{2\pi i}\oint dz T(z)\epsilon(z)\phi(w) - \frac{1}{2\pi i}\oint dz\phi(w)T(z)\epsilon(z) \\ &= \frac{1}{2\pi i}\oint_w dz R(T(z)\epsilon(z)\phi(w))\end{aligned}$$

This expansion is taking place inside a correlator, so these products must be time ordered. Here, this amounts to radial ordering, which we have denoted by $R()$. This means that in the first term, w is inside the contour of integration, while in the second, it is outside. The difference in sign between the integrals only amounts to them being taken in opposite directions. This means these integrals almost completely cancel out, leaving only the integral of the radially ordered product over a contour circling w.
So:

$$\delta\phi(w) = \frac{1}{2\pi i}\oint_w dz R(T(z)\epsilon(z)\phi(w))$$

where the integral is taken over a contour around w. However, we already have an expression for $\delta\phi$ above:

$$\delta\phi = \epsilon(w)\partial_z\phi + h\phi(w)\partial_z\epsilon$$

If we want to match these terms, we will have to expand $\epsilon(z) = \epsilon(w) + \partial_z\epsilon(w)(z-w) + ...$ and look at the residues of the integral. Thus:

$$\begin{aligned} \epsilon(w)\partial_z\phi + h\phi(w)\partial_z\epsilon &= \frac{1}{2\pi i}\oint_w dz R(T(z)(\epsilon(w) + \partial_z\epsilon(w)(z-w) + ...)\phi(w)) \\ \implies R(T(z)\phi(w)) &= \frac{h}{(z-w)^2}\phi(w) + \frac{1}{z-w}\partial_w\phi(w) + ... \end{aligned}$$

We have obtained all the singular terms in the $T\phi$ OPE, for any conformal field theory.

This can be extended easily to obtain a further result. Consider some correlator of $T(z)$ with a string of chiral primary fields:

$$\langle T(z)\phi_1(z_1)...\phi_n(z_n)\rangle$$

Performing the same conformal transformation as above, each of the ϕ_i will vary accordingly. Following the same logic, we wrap the contour around the z_i, obtaining a sum of integrals. Performing the integrals, we arrive at the conformal Ward identity:

$$\langle T(z)\phi_1(z_1)...\phi_n(z_n)\rangle = \sum_{i=1}^{n}\left(\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i}\partial_i\right)\langle\phi_1(z_1)...\phi_n(z_n)\rangle$$

5.3 The TT OPE and the Central Charge

Unfortunately, we cannot use this result to calculate the $T(z)T(w)$ OPE, as T is not a primary field, so does not transform in the necessary way. We can, however, do the following expansion (as we showed in 5 that $T(z)$ is holomorphic):

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n \tag{14}$$

where the labelling of the Laurent modes will soon become clear. Choosing $\epsilon(z) = -\epsilon_n z^{n+1}$:

$$\begin{aligned} Q_n &= \frac{1}{2\pi i}\oint dz T(z)(-\epsilon_n z^{n+1}) \\ &= -\epsilon_n L_n \end{aligned}$$

so we have that the Laurent modes of $T(z)$ are the generators of the conformal transformations, and as such must obey the Virasoro algebra.
We will use this fact to verify the following expression for the singular part of the $T(z)T(w)$ OPE:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + ...$$

Inverting 14 we get:

$$L_n = \frac{1}{2\pi i}\oint dz z^{n+1} T(z)$$

therefore

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2\pi i}\oint dz z^{m+1} \frac{1}{2\pi i}\oint dw w^{n+1}[T(w), T(z)] \\
&= -\frac{1}{2\pi i}\oint_{C(0)} dz z^{m+1} \frac{1}{2\pi i}\oint_{C(z)} dw w^{n+1} R(T(w)T(z)) \\
&= -\frac{1}{2\pi i}\oint_{C(0)} dz z^{m+1} \frac{1}{2\pi i}\oint_{C(z)} dw w^{n+1} \left(\frac{c/2}{(w-z)^4} + \frac{2T(z)}{(w-z)^2} + \frac{\partial_z T(z)}{w-z}\right)
\end{aligned}$$

with the second integral being taken around z fixed (due to the radial ordering), and the first around the origin. Note that the OPE is an expansion in $w - z$, as here we have taken $|w| > |z|$. We want the contour around z to be taken anticlockwise, so we must introduce a minus sign. Also note that for brevity, we have omitted the ellipses denoting the non-singular terms as they will note contribute.
We now pick out the residues to do the first integral. To do so, we must expand w^{n+1} as follows:

$$\begin{aligned}
w^{n+1} &= (z + w - z)^{n+1} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} z^{n+1-k}(w-z)^k
\end{aligned}$$

Using this, and taking each term separately:

$$\begin{aligned}
-\frac{1}{2\pi i}\oint_{C(0)} dz z^{m+1} \frac{1}{2\pi i}\oint_{C(z)} dw \sum_{k=0}^{n+1} \binom{n+1}{k} z^{n+1-k} \left(\frac{c/2}{(w-z)^{4-k}}\right) \\
&= -\frac{1}{2\pi i}\oint_{C(0)} dz z^{m+1} \binom{n+1}{3} z^{n-2}\frac{c}{2} \\
&= -\frac{1}{2\pi i}\oint_{C(0)} dz z^{m+n-1} \frac{(n+1)(n)(n-1)}{6}\frac{c}{2} \\
&= -\frac{c}{12}(n)(n^2-1)\delta_{m+n,0} \\
&= \frac{c}{12}(m)(m^2-1)\delta_{m+n,0}
\end{aligned}$$

Here we replaced n with $-m$ as these are equal whenever the $\delta_{m+n,0}$ is non-zero.

$$\begin{aligned}
-\frac{1}{2\pi i}\oint_{C(0)} dz z^{m+1}\frac{1}{2\pi i}\oint_{C(z)} dw \sum_{k=0}^{n+1}\binom{n+1}{k} z^{n+1-k}\left(\frac{2T(w)}{(w-z)^{2-k}}\right) \\
=&-\frac{1}{2\pi i}\oint_{C(0)} dz z^{m+1}\binom{n+1}{1} z^{n}(2T(w)) \\
=&-\frac{2(n+1)}{2\pi i}\oint_{C(0)} dz z^{m+n+1}T(w) \\
=&-2(n+1)L_{m+n}
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{2\pi i}\oint_{C(0)} dz z^{m+1}\frac{1}{2\pi i}\oint_{C(z)} dw \sum_{k=0}^{n+1}\binom{n+1}{k} z^{n+1-k}\left(\frac{\partial_w T(w)}{(w-z)^{1-k}}\right) \\
=&-\frac{1}{2\pi i}\oint_{C(0)} dz z^{m+1}\binom{n+1}{0} z^{n+1}\partial_w T(w) \\
=&-\frac{1}{2\pi i}\oint_{C(0)} dz z^{m+n+2}\partial_w T(w) \\
=&\frac{m+n+2}{2\pi i}\oint_{C(0)} dz z^{m+n+1}T(w) \\
=&(m+n+2)L_{m+n}
\end{aligned}$$

Here we integrated by parts.

Finally, we obtain the result:

$$\begin{aligned}
[L_m, L_n] &= (m+n+2)L_{m+n} - 2(n+1)L_{m+n} + \frac{c}{12}(m)(m^2-1)\delta_{m+n,0} \\
&= (m-n)L_{m+n} + \frac{c}{12}(m)(m^2-1)\delta_{m+n,0}
\end{aligned}$$

This justifies our claim, which we repeat here:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \ldots$$

This can be calculated using Wick's theorem for a given energy-momentum tensor; for example, the free boson has $c = 1$ and the free fermion has $c = \frac{1}{2}$. See section 5.3 of [3] for details.

5.4 Calculating Correlators of Descendant Fields

We now have the necessary machinery to attack the correlators of descendant fields. Descendant states are obtained from a primary state by acting on them with the L_n. We have shown that the coefficients of the Laurent modes of the energy-momentum tensor ast as the raising operators of our highest weight representation of the Virasoro algebra.

We have defined the descendant fields as those that appear in the the $T\phi$ OPE, expanded about z:

$$T(\zeta)\phi(z) = \sum_{n\in\mathbb{Z}} \frac{L_n\phi(z)}{(\zeta - z)^{n+2}}$$

Each descendant field has a clear correspondence to the descendant states found by acting of the highest weight state with the L_n. This can inverted using a contour integral to obtain an expression for $L_n\phi$.

$$L_n\phi(z) = \frac{1}{2\pi i}\oint_z d\zeta(\zeta - z)^{n+1}T(\zeta)\phi(z)$$

Notice that for L_{-2} acting on the identity field, we obtain $T(w)$ (as T is holomorphic); thus, T itself is a descendant of the identity field.

Consider the correlator of some descendant of $\phi_1(z)$ with $\phi_2(w)$ (ϕ_1 and ϕ_2 being primary). We can express L_n in terms of $T(z)$ as follows:

$$\langle (L_n\phi_1(z))\phi_2(w)\rangle = \frac{1}{2\pi i}\oint_z d\zeta(\zeta - z)^{n+1}\langle T(\zeta)\phi_1(z)\phi_2(w)\rangle$$

where the contour of integration circles z, and not w. We can expand and reverse the contour, taking it past infinity on the Riemann sphere and wrapping it around w to obtain:

$$\langle (L_n\phi_1(z))\phi_2(w)\rangle = -\frac{1}{2\pi i}\oint_w d\zeta(\zeta - z)^{n+1}\langle T(\zeta)\phi_1(z)\phi_2(w)\rangle$$

At which point we use the $T\phi$ OPE:

$$\langle (L_n\phi_1(z))\phi_2(w)\rangle = -\frac{1}{2\pi i}\oint_w d\zeta(\zeta - z)^{n+1}\left\langle \left(\frac{h_2}{(\zeta - w)^2} + \frac{1}{\zeta - w}\partial_w\right)\phi_1(z)\phi_2(w)\right\rangle$$

If instead of a single primary ϕ_2, we had a string of primaries, this would be a sum of integrals. In each, the OPE would be taken with the ϕ evaluated at the center of the contour of integration.

Now, using $(\zeta - z)^{n+1} = (w - z)^{n+1} + (n+1)(w - z)^n(\zeta - w) + ...$ and noticing that the higher order terms will not contribute:

$$\begin{aligned}
&\langle (L_n\phi_1(z))\phi_2(w)\rangle \qquad (15)\\
&= -\frac{1}{2\pi i}\oint_w d\zeta\Big((w - z)^{n+1} + (n+1)(w - z)^n(\zeta - w)\Big)\left\langle\left(\frac{h_2}{(\zeta - w)^2} + \frac{1}{\zeta - w}\partial_w\right)\phi_1(z)\phi_2(w)\right\rangle\\
&= -\frac{1}{2\pi i}\oint_w d\zeta\Big((w - z)^{n+1} + (n+1)(w - z)^n(\zeta - w)\Big)\left(\frac{h_2}{(\zeta - w)^2} + \frac{1}{\zeta - w}\partial_w\right)\langle\phi_1(z)\phi_2(w)\rangle\\
&= -\frac{1}{2\pi i}\oint_w d\zeta\left((w - z)^{n+1}\frac{1}{\zeta - w}\partial_w + (n+1)(w - z)^n\frac{h_2}{\zeta - w}\right)\langle\phi_1(z)\phi_2(w)\rangle\\
&= -\Big((w - z)^{n+1}\partial_w + (n+1)(w - z)^n h_2\Big)\langle\phi_1(z)\phi_2(w)\rangle
\end{aligned}$$

This procedure can easily be extended to some string of primary fields, instead of just ϕ_2. Instead of the differential operator $-\left((w-z)^{n+1}\partial_w + (n+1)(w-z)^n h_2\right)$ we obtain a sum of such terms, over the positions of the fields and their conformal dimension.

We can also use this procedure on a correlator of a number of descendant fields, using the TT OPE, though this calculation can quickly become quite expansive.

5.5 The $\phi\phi$ OPE

Here we shall see more of the remarkable power of the OPE: by expanding the product of two fields into a sum of local fields, we can reduce a 4-point correlator to a sum of 3-point or 2-point functions. Let us make the following ansatz as to the form of the OPE of two chiral primary fields:

$$\phi_i(z)\phi_j(w) = \sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(z-w)^{-h_i-h_j+h_k+n}\partial^n\phi_k(w)$$

where k labels the primary and quasi-primary fields, and $\partial^n\phi_k$ gives us the descendants of those fields obtained by acting with L_{-1}. We only need these descendants as we are summing over all quasi-primaries, as well as all primaries, and we can always write any other descendant as a combination of quasi-primaries and L_{-1}^n descendants. The C_{ij}^k are constants which do not depend on n, such that the a_{ijk}^n are 1 for $n=0$ (n is an index, not a power). Our guess is less arbitrary than it may appear; we can put powerful constraints on this expansion by using the differential equations 12, which we repeat here:

$$\begin{aligned} L_{-1}\phi &= \sum_i \partial_i\phi \\ L_0\phi &= \sum_i (z_i\partial_i + h_i)\phi \\ L_1\phi &= \sum_i (z_i^2\partial_i + 2h_i z_i)\phi \end{aligned}$$

Recall that the first equation encodes translational invariance. This constrains the coefficients in the expansion to only depend on $z-w$.
The second is scaling invariance, which forces us to choose the correct powers of the dimensionful quantities; $\phi_i(z)\phi_j(w)$ has scaling dimension h_i+h_j, so our expansion should too. The fields on the RHS have scaling dimension h_k+n, so the exponent of the $z-w$ factor must cancel this.
Finally, applying the final equation and enforcing invariance under SCTs will give us the form of the a_{ijk}^n. The C_{ij}^k turn out to be familiar theory dependent quantities, the 3-point function structure constants. This is found by substituting the above for a pair of fields in the 3-point correlator and using the explicit expression for the resulting sum of 2-point functions. We will not present this here, but merely mention that the identity $(1+x)^{-p} = \sum_{n=0}^{\infty}(-1)^n\binom{p+n-1}{n}x^n$ is necessary to correctly match the z_i.
It is also interesting to consider using this expansion in the 2-point function. This will result in a sum of 1-point functions, all of which will vanish except for the identity field,

which has $h = 0$. We therefore obtain:

$$\frac{\delta_{h_i,h_j}}{(z_1 - z_2)^{2h_i}} = \frac{C_{ij}^0}{(z-w)^{h_i+h_j}}$$

so $C_{ij}^0 = \delta_{h_i,h_j}$, as expected.

Before applying L_1, we will demonstrate the method with the simpler L_{-1}. The method consists of insisting that use of the OPE commutes with the application of each of the above operators. In the first instance we will apply L_{-1} to $\phi_i(z)\phi_j(w)$, and then expand using our OPE.

$$\begin{aligned}
L_{-1}(\phi_i(z)\phi_j(w)) =& (\partial_z + \partial_w)\phi_i(z)\phi_j(w) \\
=& (\partial_z + \partial_w)\left(\sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(z-w)^{-h_i-h_j+h_k+n}\partial^n\phi_k(w)\right) \\
=& \sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(-h_i-h_j+h_k+n)(z-w)^{-h_i-h_j+h_k+n-1}\partial^n\phi_k(w) \\
&- \sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(-h_i-h_j+h_k+n)(z-w)^{-h_i-h_j+h_k+n-1}\partial^n\phi_k(w) \\
&+ \sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(z-w)^{-h_i-h_j+h_k+n}\partial^{n+1}\phi_k(w) \\
=& \sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(z-w)^{-h_i-h_j+h_k+n}\partial^{n+1}\phi_k(w)
\end{aligned}$$

Now we will expand using our OPE, and then apply L_{-1}:

$$\begin{aligned}
L_{-1}(\phi_i(z)\phi_j(w)) =& L_{-1}\left(\sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(z-w)^{-h_i-h_j+h_k+n}\partial^n\phi_k(w)\right) \\
=& \sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(z-w)^{-h_i-h_j+h_k+n}L_{-1}\left(\partial^n\phi_k(w)\right) \\
=& \sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(z-w)^{-h_i-h_j+h_k+n}\partial^{n+1}\phi_k(w)
\end{aligned}$$

We see that the constraint is indeed satisfied. Now, the slightly less trivial but certainly more interesting constraint:

$$\begin{aligned} L_1(\phi_i(z)\phi_j(w)) =& (z^2\partial_z + 2h_i z + w^2\partial_w + 2h_j w)\phi_i(z)\phi_j(w) \\ =& \sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(z^2\partial_z + w^2\partial_w)(z-w)^{-h_i-h_j+h_k+n}\partial^n\phi_k(w) \\ &+ \sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(2h_i z + 2h_j w)(z-w)^{-h_i-h_j+h_k+n}\partial^n\phi_k(w) \\ &+ \sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(z-w)^{-h_i-h_j+h_k+n} w^2\partial^{n+1}\phi_k(w) \end{aligned}$$

The first term simplifies to:

$$\begin{aligned} &\sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(z^2-w^2)(-h_i-h_j+h_k+n)(z-w)^{-h_i-h_j+h_k+n-1}\partial^n\phi_k(w) \\ =& \sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(z+w)(-h_i-h_j+h_k+n)(z-w)^{-h_i-h_j+h_k+n}\partial^n\phi_k(w) \end{aligned}$$

This can be combined with the second term:

$$\begin{aligned} L_1(\phi_i(z)\phi_j(w)) =& \sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}\Big[(z+w)(-h_i-h_j+h_k+n) \\ &+ 2h_i z + 2h_j w\Big](z-w)^{-h_i-h_j+h_k+n}\partial^n\phi_k(w) \\ &+ \sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(z-w)^{-h_i-h_j+h_k+n} w^2\partial^{n+1}\phi_k(w) \end{aligned} \tag{16}$$

Now we will expand using our OPE, and then apply L_1:

$$\begin{aligned} L_1(\phi_i(z)\phi_j(w)) =& L_1\left(\sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(z-w)^{-h_i-h_j+h_k+n}\partial^n\phi_k(w)\right) \\ =& \sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(z-w)^{-h_i-h_j+h_k+n} L_1\left(\partial^n\phi_k(w)\right) \end{aligned}$$

But what is $L_1\partial^n\phi_k(w)$? We must be wary here; we need to calculate the action of L_1 on the nth descendant of $\phi_k(w)$. This amounts to calculating $L_1 L_{-1}^n\phi_k(w)$, which we can do using the same relation we used in calculating 12:

$$\begin{aligned}
L_1 L_{-1}^n \phi_k(w) &= (L_{-1}^n L_1 + 2nL_{-1}^{n-1}L_0 + (n)(n-1)L_{-1}^{n-1})\phi_k(w)\\
&= (L_{-1}^n(w^2\partial + 2h_k w) + 2nL_{-1}^{n-1}(h_k + w\partial) + (n)(n-1)\partial^{n-1})\phi_k(w)\\
&= ((w^2\partial + 2h_k w)L_{-1}^n + 2n(h_k + w\partial)L_{-1}^{n-1} + (n)(n-1)\partial^{n-1})\phi_k(w)\\
&= ((w^2\partial + 2h_k w)\partial^n + 2n(h_k + w\partial)\partial^{n-1} + (n)(n-1)\partial^{n-1})\phi_k(w)\\
&= (w^2\partial^{n+1} + 2(h_k + n)w\partial^n + (2nh_k + n^2 - n)\partial^{n-1})\phi_k(w)
\end{aligned}$$

We can immediately note that the w^2 term here matches the w^2 term in 16 exactly. Matching the remainder of the terms, we approach the final stretch; in the following, the ∂^{n-1} term on the RHS should only be summed for $n \geq 1$, but for ease of notation we leave this implicit.

$$\begin{aligned}
&\sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(z-w)^{-h_i-h_j+h_k+n}\Big((z+w)(-h_i-h_j+h_k+n) + 2h_i z + 2h_j w\Big)\partial^n\phi_k(w)\\
=&\sum_{k,n\geq 0} \frac{C_{ij}^k a_{ijk}^n}{n!}(z-w)^{-h_i-h_j+h_k+n}\Big(2(h_k+n)w\partial^n + (2nh_k + n^2 - n)\partial^{n-1}\Big)\phi_k(w)
\end{aligned}$$

This must be true for each ϕ_k individually, so:

$$\begin{aligned}
&\sum_{n\geq 0} \frac{a_{ijk}^n}{n!}(z-w)^n\Big((z+w)(h_k+n) + (h_i - h_j)(z-w)\Big)\partial^n\phi_k(w)\\
=&\sum_{n\geq 0} \frac{a_{ijk}^n}{n!}(z-w)^n\Big(2(h_k+n)w\partial^n + (2nh_k + n^2 - n)\partial^{n-1}\Big)\phi_k(w)
\end{aligned}$$

which is equivalent to:

$$\begin{aligned}
&\sum_{n\geq 0} \frac{a_{ijk}^n}{n!}(z-w)^n(z-w)(h_k + n + h_i - h_j)\partial^n\phi_k(w)\\
=&\sum_{n\geq 1} \frac{a_{ijk}^n}{n!}(z-w)^n(2nh_k + n^2 - n)\partial^{n-1}\phi_k(w)
\end{aligned}$$

Now, reindexing:

$$\begin{aligned}
&\sum_{n\geq 0} \frac{a_{ijk}^n}{n!}(z-w)^{n+1}(h_k + n + h_i - h_j)\partial^n\phi_k(w)\\
=&\sum_{n\geq 0} \frac{a_{ijk}^{n+1}}{(n+1)!}(z-w)^{n+1}(2(n+1)h_k + (n+1)^2 - n - 1)\partial^n\phi_k(w)
\end{aligned}$$

We can now, finally, match terms:

$$\begin{aligned}
a_{ijk}^n(h_k + n + h_i - h_j) &= \frac{a_{ijk}^{n+1}}{n+1}(2(n+1)h_k + n(n+1))\\
\implies a_{ijk}^n(h_k + n + h_i - h_j) &= a_{ijk}^{n+1}(2h_k + n)\\
\implies \qquad a_{ijk}^{n+1} &= \left(\frac{h_i - h_j + h_k + n}{2h_k + n}\right) a_{ijk}^n
\end{aligned}$$

The solution of this, given that $a^0_{ijk} = 1$ is

$$a^n_{ijk} = \frac{(h_k + h_i - h_j)_n}{(2h_k)_n}$$

where $(a)_n = (a)(a+1)...(a+n-1)$ is the Pochhammer symbol.

5.6 Constraints From Null Vectors

Let us return for a moment to representation theory, to see what these results imply. Say we have a theory with some central charge c, and some primary field ϕ with weight $h = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right)$. Then, as shown in 3.3 there exists a null vector of weight 2:

$$|\chi\rangle = \left(-\frac{4h+2}{3}L_{-2} + L_{-1}^2\right)|h\rangle$$

What does this mean in terms of our fields? We know that this vector has a vanishing inner product with every other vector in the representation. For our fields, this implies that the correlator of a null vector with any number of other fields will vanish. Consider, then, the 3-point function of χ with two primaries:

$$\langle\chi(z)\phi_1(z_1)\phi_2(z_2)\rangle$$

This must vanish, as χ is null. However, as χ is a descendant of a primary, we can use the relation found in 5.4 to express this correlator as a differential operator acting on the correlator of primaries. Thus, we must have that:

$$\left(-\frac{4h+2}{3}\sum_{i=1}^{2}\left((z-z_i)^{-1}\partial_i + (z-z_i)^{-2}h_i\right) + \partial_z^2\right)\langle\phi(z)\phi_1(z_1)\phi_2(z_2)\rangle = 0$$

If we use the explicit form of the 3-point function, after some quite tedious calculation we arrive at a relation between h, h_1 and h_2:

$$(4h+2)(h + 2h_2 - h_1) = 3(h - h_1 + h_2)(h - h_1 + h_2 + 1)$$
$$\implies h_2 = \frac{2h + 6h_1 + 1}{6} \pm \frac{1}{6}\sqrt{(2h + 6h_1 + 1)^2 + 12(h^2 - 3h_1^2 + 2hh_1 + h_1 - h)}$$

The 3-point function of any triplet of fields not obeying this constraint must vanish. But, as we saw above, the structure constants of the 3-point functions enter into the $\phi\phi$ OPE. So, if h_2 is not of the above form, it cannot appear in the $\phi\phi_1$ OPE. The existence of a null vector places new constraints on the structure constants, and therefore restricts the fields which may enter into the OPE. We can place further restrictions by condsidering that the order of the fields we expand should not matter; the same fields should appear in $\phi\phi_1$ as in $\phi_1\phi$, so fields ruled out in either are ruled out in both.

These are important results, and can be pushed further; in the case of the minimal models, these rules as to which fields may appear in the OPE are enough so show that we can have a theory with finitely many fields, which is closed under expansion in an OPE. This process is known as fusion, and the rules derived from the existence of null vectors are known as fusion rules.

6 The 4-Point Correlator

6.1 4-point Correlator as a Series

Let us now use the $\phi\phi$ expansion as a tool to attack the 4-point function. We will examine the case of four identical fields ϕ of weight h:

$$\langle \phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\rangle$$

To do this, we will expand the first pair of fields and the second pair of fields, then use the explicit form of the 2-point function on the result. Our OPE will involve descendants of the field, so to begin we will work out the correlator of two descendants. Recalling 13 and 15:

$$\begin{aligned}
\langle \phi_i(z)\phi_j(w)\rangle &= \frac{\delta_{h_i,h_j}}{(z-w)^{2h_i}} \\
\implies \langle \partial_z^n\phi_i(z)\partial_w^m\phi_j(w)\rangle &= \partial_z^n\partial_w^m\frac{\delta_{h_i,h_j}}{(z-w)^{2h_i}} \\
\implies \langle \partial_z^n\phi_i(z)\partial_w^m\phi_j(w)\rangle &= (-1)^n\frac{\delta_{h_i,h_j}}{(z-w)^{2h_i+m+n}}(2h_i)_{m+n}
\end{aligned}$$

Now, the OPE for identical fields, defining $C_k \equiv C_{\phi\phi}^k$ and $a_k^n \equiv a_{\phi\phi k}^n$:

$$\phi(z)\phi(w) = \sum_{k,n\geq 0}\frac{C_k a_k^n}{n!}(z-w)^{-2h+h_k+n}\partial^n\phi_k(w)$$

So:

$$\begin{aligned}
&\langle \phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\rangle = \\
&\quad\left\langle \sum_{k,n\geq 0}\frac{C_k a_k^n}{n!}(z_{12})^{-2h+h_k+n}\partial_2^n\phi_k(z_2)\sum_{l,m\geq 0}\frac{C_l a_l^m}{m!}(z_{34})^{-2h+h_l+m}\partial_4^m\phi_l(z_4)\right\rangle \\
&= \sum_{k,n\geq 0}\frac{C_k a_k^n}{n!}(z_{12})^{-2h+h_k+n}\sum_{l,m\geq 0}\frac{C_l a_l^m}{m!}(z_{34})^{-2h+h_l+m}\langle \partial_2^n\phi_k(z_2)\partial_4^m\phi_l(z_4)\rangle \\
&= \sum_{k,n\geq 0}\frac{C_k a_k^n}{n!}(z_{12})^{-2h+h_k+n}\sum_{l,m\geq 0}\frac{C_l a_l^m}{m!}(z_{34})^{-2h+h_l+m}\frac{(-1)^n\delta_{h_l,h_k}}{(z_{24})^{2h_k+m+n}}(2h_k)_{m+n} \\
&= \frac{1}{(z_{12}z_{34})^{2h}}\sum_{k,n\geq 0}\sum_{m\geq 0}\left(\frac{z_{12}z_{34}}{z_{24}^2}\right)^{h_k}\frac{C_k^2 a_k^n a_k^m}{n!m!}z_{12}^n z_{34}^m\frac{(-1)^n}{z_{24}^{m+n}}(2h_k)_{m+n}
\end{aligned}$$

While this expression manifestly has the correct behaviour under translations, rotations and dilations it is not at all obvious that it is fully conformally invariant. In the next section we will attempt to sum part of this series, to obtain an explicit dependence on the cross-ratio.

6.2 Summing the Series: The Hypergeometric Function

Consider the terms multiplying the structure constants:

$$\sum_{n\geq 0}\sum_{m\geq 0}\left(\frac{z_{12}z_{34}}{z_{24}^2}\right)^{h_k}\frac{(h_k)_n}{(2h_k)_n}\frac{(h_k)_m}{(2h_k)_m}\frac{(2h_k)_{m+n}}{n!m!}z_{12}^n z_{34}^m\frac{(-1)^n}{z_{24}^{m+n}} \tag{17}$$

We will use the identity (proven by Adam Keilthy):

$$\frac{(a+b)_n(a+c)_n}{(a+b+c)_n}=\sum_{r=0}^{n}{}_nC_r\frac{(a)_{n-r}(b)_r(c)_r}{(a+b+c)_r} \tag{18}$$

with $a=h_k+m$, $b=-m$ and $c=h_k$.

To prove this identity we will need to use the Euler transformation for the hypergeometric function. The hypergeometric function is defined as:

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \tag{19}$$

The Euler transformation is as follows:

$${}_2F_1(u,v;w;z)=(1-z)^{w-u-v}{}_2F_1(w-u,w-v;w;z) \tag{20}$$

Finally, we will also need the generating function for the Pochhammer symbol:

$$\frac{1}{(1-z)^a}=\sum_{n=0}^{\infty}(a)_n\frac{z^n}{n!}$$

We begin by multiplying the LHS of 18 by $\frac{z^n}{n!}$ and summing over all n:

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(a+b)_n(a+c)_n}{(a+b+c)_n}\frac{z^n}{n!} &= {}_2F_1(a+b,a+c;a+b+c;z)\\
&=(1-z)^{-a}{}_2F_1(c,b;a+b+c;z)\\
&=\sum_{n=0}^{\infty}(a)_n\frac{z^n}{n!}\sum_{r=0}^{\infty}\frac{(b)_r(c)_r}{(a+b+c)_r}\frac{z^r}{r!}\\
&=\sum_{r=0}^{\infty}\frac{(b)_r(c)_r}{(a+b+c)_r}\frac{z^r}{r!}\sum_{n=r}^{\infty}(a)_{n-r}\frac{z^{n-r}}{(n-r)!}\\
&=\sum_{n=0}^{\infty}\sum_{r=0}^{n}{}_nC_r\frac{(a)_{n-r}(b)_r(c)_r}{(a+b+c)_r}\frac{z^n}{n!}
\end{aligned}$$

We can differentiate each side multiple times with respect to z, then set z to 0, and thus prove 18 for any n.

In our case, with $a = h_k + m$, $b = -m$ and $c = h_k$, we find:

$$\frac{(h_k)_n(2h_k+m)_n}{(2h_k)_n} = \sum_{r=0}^{min(m,n)} {}_nC_r \frac{(h_k+m)_{n-r}(-m)_r(h_k)_r}{(2h_k)_r}$$

Using that $(2h_k+m)_n = \frac{(2h_k)_{m+n}}{(2h_k)_m}$, and similar manipulations:

$$\frac{(h_k)_n(h_k)_m(2h_k)_{m+n}}{(2h_k)_n(2h_k)_m n!m!} = \sum_{r=0}^{min(m,n)} \frac{(h_k)_{m+n-r}(h_k)_r(-1)^r}{(2h_k)_r r!(n-r)!(m-r)!}$$

Inserting this in 17:

$$\begin{aligned}
&\sum_{n\geq 0}\sum_{m\geq 0}\left(\frac{z_{12}z_{34}}{z_{24}^2}\right)^{h_k}\sum_{r=0}^{min(m,n)} \frac{(h_k)_{m+n-r}(h_k)_r(-1)^r}{(2h_k)_r r!(n-r)!(m-r)!} z_{12}^n z_{34}^m \frac{(-1)^n}{z_{24}^{m+n}} \\
&= \left(\frac{z_{12}z_{34}}{z_{24}^2}\right)^{h_k}\sum_{r\geq 0}\sum_{n\geq r}\sum_{m\geq r}\frac{(-1)^r}{r!}\frac{(h_k)_{m+n-r}(h_k)_r}{(2h_k)_r(n-r)!(m-r)!} z_{12}^n z_{34}^m \frac{(-1)^n}{z_{24}^{m+n}} \\
&= \left(\frac{z_{12}z_{34}}{z_{24}^2}\right)^{h_k}\sum_{r\geq 0}\sum_{n\geq 0}\sum_{m\geq 0}\frac{(-1)^r}{r!}\frac{(h_k)_{m+n+r}(h_k)_r}{(2h_k)_r n!m!} z_{12}^{n+r} z_{34}^{m+r} \frac{(-1)^{n+r}}{z_{24}^{m+n+2r}} \\
&= \sum_{r\geq 0}\left(\frac{z_{12}z_{34}}{z_{24}^2}\right)^{h_k+r}\frac{(h_k)_r}{r!(2h_k)_r}\sum_{n\geq 0}\frac{(-1)^n}{n!}\left(\frac{z_{12}}{z_{24}}\right)^n\sum_{m\geq 0}\frac{(h_k)_{m+n+r}}{m!}\left(\frac{z_{34}}{z_{24}}\right)^m \\
&= \sum_{r\geq 0}\left(\frac{z_{12}z_{34}}{z_{24}^2}\right)^{h_k+r}\frac{(h_k)_r}{r!(2h_k)_r}\sum_{n\geq 0}\frac{(-1)^n(h_k)_{n+r}}{n!}\left(\frac{z_{12}}{z_{24}}\right)^n\sum_{m\geq 0}\frac{(h_k+n+r)_m}{m!}\left(\frac{z_{34}}{z_{24}}\right)^m
\end{aligned}$$

Recognising the generating function for the Pochhammer we get:

$$
\begin{aligned}
&= \sum_{r\geq 0} \left(\frac{z_{12}z_{34}}{z_{24}^2}\right)^{h_k+r} \frac{(h_k)_r}{r!(2h_k)_r} \sum_{n\geq 0} \frac{(-1)^n (h_k)_{n+r}}{n!} \left(\frac{z_{12}}{z_{24}}\right)^n \left(1-\left(\frac{z_{34}}{z_{24}}\right)\right)^{-h_k-n-r} \\
&= \sum_{r\geq 0} \left(\frac{z_{12}z_{34}}{z_{24}(z_{24}-z_{34})}\right)^{h_k+r} \frac{(h_k)_r}{r!(2h_k)_r} \sum_{n\geq 0} \frac{(-1)^n (h_k)_{n+r}}{n!} \left(\frac{z_{12}}{z_{24}}\right)^n \left(\frac{z_{24}}{z_{24}-z_{34}}\right)^n \\
&= \sum_{r\geq 0} \left(\frac{z_{12}z_{34}}{z_{24}z_{23}}\right)^{h_k+r} \frac{(h_k)_r}{r!(2h_k)_r} \sum_{n\geq 0} \frac{(-1)^n (h_k)_{n+r}}{n!} \left(\frac{z_{12}}{z_{24}}\right)^n \left(\frac{z_{24}}{z_{23}}\right)^n \\
&= \sum_{r\geq 0} \left(\frac{z_{12}z_{34}}{z_{24}z_{23}}\right)^{h_k+r} \frac{(h_k)_r (h_k)_r}{r!(2h_k)_r} \sum_{n\geq 0} \frac{(-1)^n (h_k+r)_n}{n!} \left(\frac{z_{12}}{z_{24}}\right)^n \left(\frac{z_{24}}{z_{23}}\right)^n \\
&= \sum_{r\geq 0} \left(\frac{z_{12}z_{34}}{z_{24}z_{23}}\right)^{h_k+r} \frac{(h_k)_r (h_k)_r}{r!(2h_k)_r} \left(1+\frac{z_{12}}{z_{23}}\right)^{-h_k-r} \\
&= \sum_{r\geq 0} \left(\frac{z_{12}z_{34}}{z_{24}z_{23}}\right)^{h_k+r} \frac{(h_k)_r (h_k)_r}{r!(2h_k)_r} \left(\frac{z_{23}}{z_{13}}\right)^{h_k+r} \\
&= \sum_{r\geq 0} \left(\frac{z_{12}z_{34}}{z_{24}z_{13}}\right)^{h_k+r} \frac{(h_k)_r (h_k)_r}{r!(2h_k)_r}
\end{aligned}
$$

We recognise this coefficient from the definition of the hypergeometic function ${}_2F_1(h_k, h_k; 2h_k; u)$. This matches the result in the literature:

$$\langle \phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\rangle = \frac{1}{(z_{12}z_{34})^{2h}} \sum_{k\geq 0} C_k^2 {}_2F_1(h_k, h_k; 2h_k; u)$$

This is a function of only the cross-ratio $u = \frac{z_{12}z_{34}}{z_{13}z_{24}}$ and so manifestly conformally invariant.

6.3 A New Constraint: Crossing Symmetry

Let us examine this expression further.

$$\langle \phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\rangle = \frac{1}{(z_{12}z_{34})^{2h}} \sum_{k\geq 0} C_k^2 {}_2F_1(h_k, h_k; 2h_k; u)$$

The ${}_2F_1(h_k, h_k; 2h_k; u)$ terms are known as the conformal blocks. It is remarkable that the dependence of the 4-point correlator on the conformal dimension of the summed primary fields reduces to a known function, multiplied by some constants. In fact, as the C_k are the structure constants from the 3-point correlator, we know they must be real; we know, therefore, that C_k^2 is positive.

We stated that the OPE algebra must be associative, but we have not yet used this fact. Above, we used the OPE of the first pair and the last pair of fields; we could instead have used the OPE of ϕ_2 with ϕ_3 first. This is known as crossing symmetry. In terms of the resulting expression, this amounts to swapping ϕ_2 with ϕ_4. The new cross ratio is then:

$$v = \frac{z_{14}z_{32}}{z_{13}z_{42}}$$

However, we can notice:

$$\begin{aligned}
1 - v &= 1 - \frac{z_{14}z_{32}}{z_{13}z_{42}} \\
&= \frac{z_{13}z_{42} - z_{14}z_{32}}{z_{13}z_{42}} \\
&= \frac{(z_1z_4 - z_1z_2 - z_3z_4 + z_3z_2) - (z_1z_3 - z_1z_2 - z_4z_3 + z_4z_2)}{z_{13}z_{42}} \\
&= \frac{z_1z_4 + z_3z_2 - z_1z_3 - z_4z_2}{z_{13}z_{42}} \\
&= \frac{z_{12}z_{43}}{z_{13}z_{42}} \\
&= \frac{z_{12}z_{34}}{z_{13}z_{24}} \\
&= u
\end{aligned}$$

so we obtain an interesting constraint on the structure constants:

$$\begin{aligned}
&\frac{1}{(z_{12}z_{34})^{2h}} \sum_{k\geq 0} C_k^2 {}_2F_1(h_k, h_k; 2h_k; u) = \frac{1}{(z_{14}z_{32})^{2h}} \sum_{k\geq 0} C_k^2 {}_2F_1(h_k, h_k; 2h_k; 1-u) \\
\iff &\frac{(z_{13}z_{24})^{2h}}{(z_{12}z_{34})^{2h}} \sum_{k\geq 0} C_k^2 {}_2F_1(h_k, h_k; 2h_k; u) = \frac{(z_{13}z_{24})^{2h}}{(z_{14}z_{32})^{2h}} \sum_{k\geq 0} C_k^2 {}_2F_1(h_k, h_k; 2h_k; 1-u) \\
\iff &(1-u)^{2h} \sum_{k\geq 0} C_k^2 {}_2F_1(h_k, h_k; 2h_k; u) = (u)^{2h} \sum_{k\geq 0} C_k^2 {}_2F_1(h_k, h_k; 2h_k; 1-u) \\
\iff &\sum_{k\geq 0} C_k^2 \left((1-u)^{2h} {}_2F_1(h_k, h_k; 2h_k; u) - u^{2h} {}_2F_1(h_k, h_k; 2h_k; 1-u)\right) = 0
\end{aligned}$$

This is a continuously infinite set of equations, with associativity of the OPE built in. These equations are of a form that allows information about possible unitary conformal field theories to be obtained, using methods of linear programming; see [5] for a good introduction to this. The method as a whole is known as the conformal bootstrap; see [4] for an introduction.

7 Further Research

From this point, there are many avenues of further exploration. Using the linear programming methods mentioned above, one can find bounds for operator dimensions. For example, given the operator of smallest dimension in the theory, an upper bound can be found for the dimension of the operator of second smallest dimension. Interestingly, this bound shows a kink at the dimension of the σ field of the critical Ising model.
The bounds can be improved by considering other correlators; above we only considered the 4-point function of identical fields, but conformal blocks can be found for mixed correlators too, and new, stronger bounds derived from this. Using these bounds one can look for more kinks, or other phenomena (like minima in the OPE coefficients) which occur at the Ising model, and the other minimal models. One can then extend this to higher dimensions, looking for similar phenomena in results obtained by applying conformal bootstrap methods, hinting at possible unitary conformal field theories.

8 Bibliography

References

[1] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. Nuclear Physics B, 241(2):333–380, July 1984.

[2] Ralph Blumenhagen and Erik Plauschinn. Basics in Conformal Field Theory. In Introduction to Conformal Field Theory, number 779 in Lecture Notes in Physics, pages 5–86. Springer Berlin Heidelberg, 2009. DOI: 10.1007/978-3-642-00450-6_2.

[3] Philippe Di Francesco, Pierre Mathieu, and David Sénéchal. Conformal Field Theory. Graduate Texts in Contemporary Physics. Springer New York, New York, NY, 1997.

[4] David C. Lewellen. Constraints for Conformal Field Theories on the Plane: Reviving the Conformal Bootstrap. Nucl.Phys., B320:345, 1989.

[5] Miguel F. Paulos. JuliBootS: a hands-on guide to the conformal bootstrap. arXiv:1412.4127 [cond-mat, physics:hep-th], December 2014. arXiv: 1412.4127.

[6] David Tong. Lectures on String Theory. arXiv:0908.0333 [hep-th], August 2009. arXiv: 0908.0333.