2D CFT: Representation Theory and The Conformal Bootstrap

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Introduction

What is a conformal field theory?
A quantum field theory in which, instead of looking at an action, we examine how things change under ***conformal transformations***.

Why should we care?
It's powerful; these considerations let us ***fix the forms of the correlators*** and (through unitarity, which we want for sensible physics) ***restrict the possible structure constants***. It can even, in two dimensions, result in something very rare, and certainly worth investigating: an ***exactly solvable*** quantum field theory.

Conformal Transformations

What links these transformations?
They are ***angle preserving***, but can warp lines and distances arbitrarily. Forcing our theory to be invariant under these transformations means ***the physics doesn't care what scale we observe it at.*** Imagine quantum mechanics, or galactic dynamics, applying at every scale!

Our theory should be invariant under:

- The usual Poincaré transformations: **translations, rotations and boosts**.
- **Dilations** of the coordinates.
- The **special conformal transformations** (SCTs).

An SCT is an inversion of the coordinates, followed by a translation, followed by another inversion.

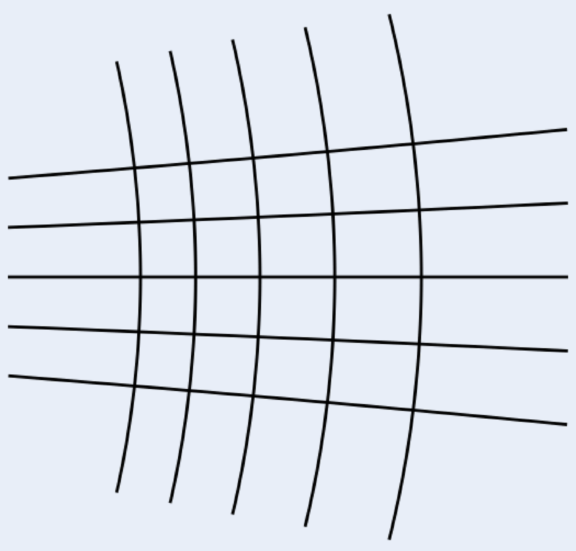

Figure 1: Image of a grid under an SCT. Note that the lines remain perpendicular.

Do any systems actually satisfy this?
The ***Ising model*** at its ***critical point*** has the same fluctuations at all length scales, as it passes from the low-order to high-order phases.

Why is 2D special?
The meromorphic functions are locally conformal. This means we can use the structure of their generators, the ***Witt algebra***, and its central extension, the ***Virasoro algebra***. It is this structure which allows for an exact solution, letting us calculate all the correlators between all the fields.

References

[1] A. Belavin et al. Infinite conformal symmetry in two-dimensional quantum field theory. Nuclear Physics B, 241(2):333–380, July 1984.

[2] P. Di Francesco, P. Mathieu, and D. Sénéchal. Conformal Field Theory. Springer New York, 1997.

[3] M. F. Paulos. JuliBootS. arXiv:1412.4127 [cond-mat, physics:hep-th], December 2014. arXiv: 1412.4127.

Fields and their Products

We use complex coordinates, z and $\bar{z}$ (though in the following only z dependence is explicit).

Primary fields: Under a conformal transformation primary fields transform with a factor of $\left(\frac{\partial f}{\partial z}\right)^h$, where h is the *conformal weight* of the field. This property tells us how the correlators of these fields transform; from this, we can deduce their explicit forms. For example, the 2-point function:

$$\langle \phi_1(z)\phi_2(w)\rangle = \frac{\delta_{h_1,h_2}}{(z-w)^{2h_1}}$$

- Both sides are manifestly translation invariant.
- $h_1 = h_2$ ensures that both sides transform the same way under inversions.
- Both sides scale with λ^{-2h_1} (dilation: $\lambda \in \mathbb{R}$, $\lambda > 0$, rotation $\lambda = e^{i\theta}$).

Operator Product Expansion (OPE): We assume a product of fields (inside a correlator) can be written as a sum of local fields. Very useful; we use it to go from n-point to (n-1)-point correlators.

Using the OPE

Energy-Momentum Tensor: In a 2D CFT, there are only two components, $T(z)$ and $\bar{T}(\bar{z})$. We can expand in Laurent modes which, due to their relation to the symmetry generators, obey the *Virasoro algebra* (with *central charge*, c, theory dependent):

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n, \qquad [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0} \tag{1}$$

Descendant Fields: The descendant fields are those that appear in the OPE of a primary field with the energy-momentum tensor. Using 1 the correlator of a descendant $L_n\phi_1$ with a primary ϕ_2 can be found in terms of the correlator of primaries (see [1] for extensions):

$$\langle (L_n\phi_1(w))\phi_2(z)\rangle = -\left((z-w)^{n+1}\partial_z + (n+1)(z-w)^n h_2\right)\langle \phi_1(w)\phi_2(z)\rangle$$

Solving the Theory

What if the OPE forces us to consider ***infinitely many fields***? Only certain conformal weights and central charges are consistent with finitely many primary fields. Eg. 2D critical Ising model ($c = \frac{1}{2}$): the identity ($h_1 = 0$), the spin field σ ($h_\sigma = \frac{1}{16}$) and the energy field ϵ ($h_\epsilon = \frac{1}{2}$). These ***closed theories*** are known as the *minimal models*; we have explicit formulas for them due to Kac:

$$c = 1 - \frac{6}{m(m+1)}, \qquad h_{r,s} = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}$$

with $1 \leq r < m$, $1 \leq s < r$ and m an integer greater than 2 [2]. This result of Kac comes from considering certain objects known as *singular vectors*; they are descendants that act like new primary states. ***These objects can consistently be set to zero***, causing all their descendants to vanish; in the minimal models, this leaves only finitely many fields.

The Conformal Bootstrap

The above results were obtained by unitarity arguments using the Virasoro algebra; powerful, but restricted to 2D. Results in other dimensions can be obtained by enforcing ***associativity of the OPE***, by a numerical procedure known as the conformal bootstrap [3]. Given a field of lowest dimension, this method places an ***upper bound on the dimension of the next lowest field***.

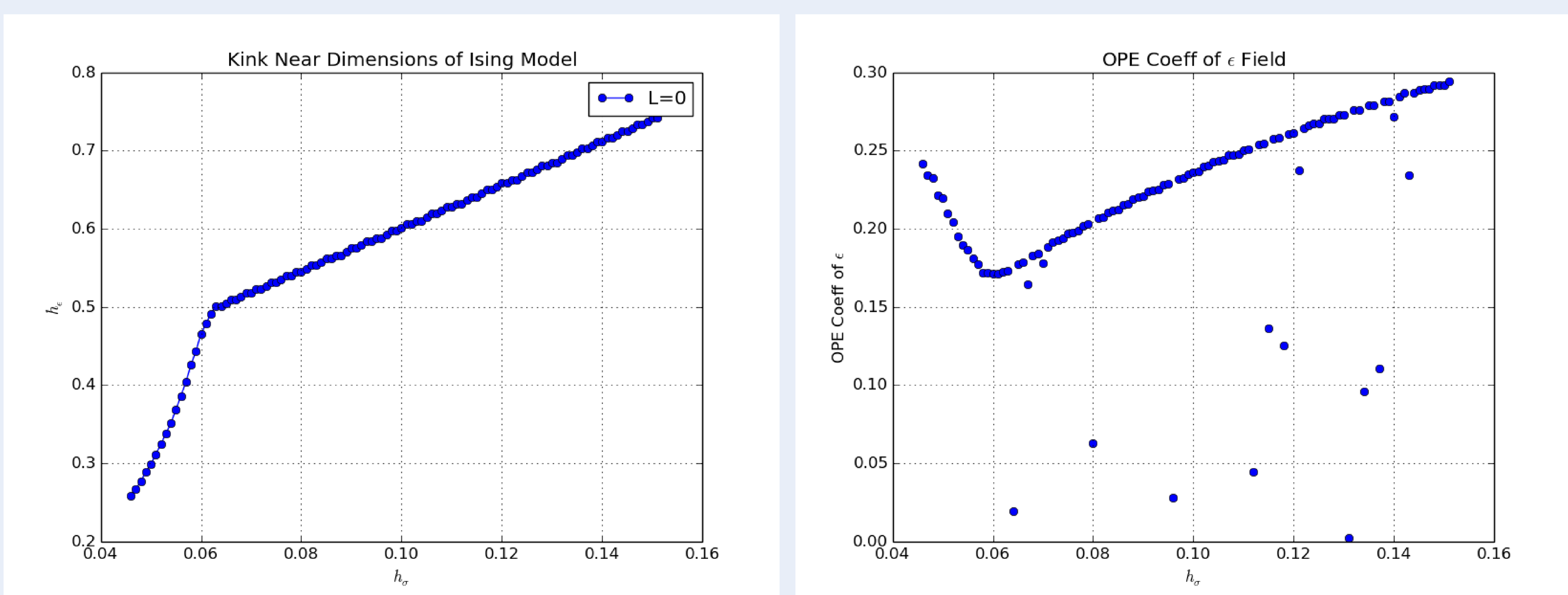

Figure 2: h_ϵ upper bound kink (left); ϵ OPE coeff minimum (right)

Interestingly, the graph of this upper bound shows a kink at the dimensions of the Ising model, and the graph of the OPE coefficient of this field shows a minimum. By studying how these graphs behave around exactly known solutions in 2D (the Ising model and the other minimal models), it is possible we will learn more about the possible conformal dimensions of higher dimensional theories.